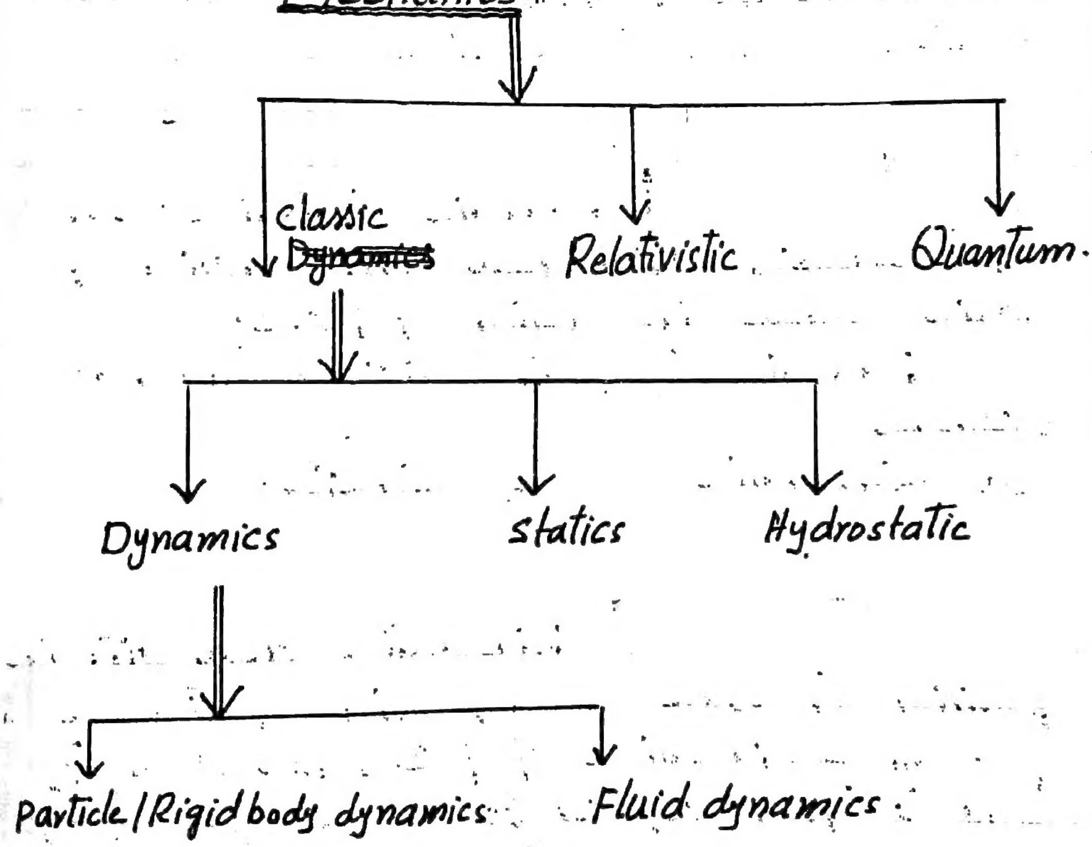
Mechanics- P- I

MECHANICS

Outlines of Syllabus

General motion of a rigid body, Euler's Theorem and Chasles' Theorem, Euler's angles, Moments and products of Inertia, Inertia tensor, Principal axes and principal Moments of inertia, Kinetic energy and Angular Momentum of a rigid body, Momental ellipsoid and equimomental systems, Euler's dynamical equations and their solution in special cases. Heavy symmetrical top, equilibrium of a rigid body, general conditions of equilibrium and deduction of conditions in special cases.

Mechanics



Mechanics

Mechanics is the science concerning to the motion and rest and laws which govern these states of bodies or particles.

Classical Mechanics

That branch of mechanics which deals with macroscopic objects.

Relativistic Mechanics

In this branch of mechanics the velocity of light is considered absolute and all velocities are taken relative to the velocity of light.

Quantum Mechanics

It deals with the microscopic particles or bodies (fundamental particles or elementry particles)

Dynamics

Dynamics is that branch of Mechanics which deals with motion of bodies under the action of forces.

Dynamics is further divided into two branches

(a) kinematics (b) Kinetics.

Kinematics

Kinematics deals with the position in space as a function of time apart from all considerations of force, mass or energy and is often referred to as the "geometry of

of motion. The calculation of flight trajectories of aircraft, spaceships, rockets are examples of kinematical problems.

Kinetics#

That branch of dynamics which deals with effects of forces on the motion of bodies and links or relates the action of forces on bodies to their resulting motion.

Remarks# The functions involved in dynamical problems are (for the most part) differential Co-efficients w.r.t time (t) as independent variable. The formulation of a dynamical problem in general consists of one or more relations between certain variables (Co-ordinates of position) and their differential Co-efficients w.r.t time. such relations are called differential equations.

Particle#

A body with negligible size so that it may be described by a geometrical point is called particle.

OR

A very very small mass concentrated at a point is called particle.

OR

A very small mass which can not be further divided

OR

In mathematical sense a particle is a body whose dimensions approach zero so that it may be analysed as a point mass.

Frequentally a particle is chosen as a differential element of a body. Also when the dimensions of a body are irrelevant to the description of its position or its motion, the body may be treated as particle.

Any body whose size is very small as compared to other bodies can be regarded as a particle.

Rigid Body

A body is said to be rigid when it has no relative deformation between its parts under external forces. In other words the distance between the particles of a rigid body doesnot change.

In real world no rigid body exist and therefore the concept of rigid body is an ideal concept.

All real objects undergo some changes in shape when they are subjected to forces. When such changes in shape are negligible as compared to over all dimension of body or with the changes in the position of body, then Then body may be considered as a rigid body.

Remarks # 1) When a body moves all parts of the body have not necessarily the same motion but if the body is very small, then the differences between the motions of its different parts are unimportant.

(2) In order to describe the motion of body or of a point two things are needed.

(a) a frame of reference (b) a time-keeper.

Reference Frame

Position in space is determined relative some geometric reference system by means of linear and angular measurements.

The basic frame of reference for laws of Newtonian mechanics is the primary inertial system or astronomical frame of reference which is an imaginary set of rectangular axes assumed to have no translation or rotation in space. Measurements show that laws of Newtonian mechanics are valid for this reference system as long as any velocities involved are neglegible compared with speed of light.

The inertial frame is required i.e frame be such that a free particle in it continues to be in state of rest or of uniform motion along a straight line. The reason for employing inertial frames is that physical laws remain unaffected by the choice of the frame of reference.

The search for such frame, call it the absolute frame, w.r.t which other inertial frames are to be defined, is, therefore important.

Existence of an Absolute Frame

Consider a frame of reference fixed on the surface of the earth. Experiments show that most of the physical phenomena can be adequately well explained with reference to this frame. But if we look a little more closely, there are some discrepencies between the predictions made in this frame and the actual observations. and a correction to the basic equations of mechanics

must be applied for measurements made relative to the earth's reference frame. For example, assume that a body is falling under the effect of gravity from a hight which is small compared with the radius of earth, so that the acc. due to gravity may be taken as constant. The usual equation of motion indicates a vertical fall of the body. It also appears so. But careful observation shows that there is a slight deviation from the vertical to right of its course. The curved wind shows the same phenomenon more clearly. Likewise, a body on which no external force is acting appears to move in the earth's frame of reference with constant speed along a straight line. Careful measurements, however would show that trajectory is not really a straight line but slightly curved. These considerations show that the frame fixed to earth's surface is not absolute because of the earth's rotation.

Consider now a frame of reference located at the centre of mass of earth and fixed orientation w.r.t to the distant stars which may be assumed to at rest to a very good approximation. The frame fixed to the earth surface is now rotating w.r.t this frame with the angular velocity of earth. Computation of the trajectory of a body falling freely under gravity near the surface of earth now shows the deviation to its right correctly. This frame is therefore nearer to the inertial frame than the frame attached to the surface. It is not, however, an inertial frame since it does not explain correctly the tidal effect of the moon. The analysis of earth-moon system in this frame indicates that there will be only one tide in every 24 hours at any

, whereas actually two tides are observed every day. This discrepancy is caused by the orbital motion of the earth. theoretically, the frame attached to the centre of the universe (if we can define one) may be answer to our search for an absolute frame.

Inertial Frame of Reference

An inertial frame of reference is the frame in which Newtons Laws remain valid. Because Newton's 1st and 3rd Laws can be derived from the 2nd Law, we can define an inertial frame a frame in which 2nd Law of motion remains valid.

Note # (1) Rotating frames are not inertial frame.

(2) In a mathematical model, we may employ a rigid body as the frame of reference. As we may introduce any number of rigid bodies moving relative to one another, therefore we have thus at our disposal any number of frame of reference.

Time

Time is a measure of the succession of events and is considered as absolute quantity in Newtonian mechanics.

Note # It is important to realize that there is no such thing as absolute time. but the period of rotation of the Earth relative to the fixed stars provides a unit of time, the sidereal day, which so far as it can be tested with other time measures is constant and therefore adequate for the purposes of ordinary dynamics.

Space# space is the geometric region occupied by the bodies.

Particle Dynamics

Although our topic is rigid body dynamics but since the rigid body is considered as made of particles which are closely packed and distance between which remain fixed, therefore it is important to discuss some basic principals of particle dynamics.

Newton's 1st Law#

A particle subject to no external force is either at rest or moves with uniform velocity.

Newton's 2nd Law#

The acceleration of particle is proportional to the resultant force acting on it and is in the direction of this force

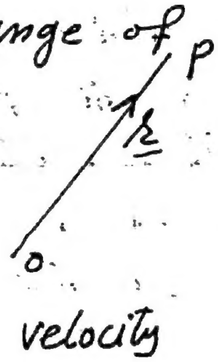
$$\underline{F} = m\underline{a}$$

Newton's 3rd Law#

The forces of action and reaction between interacting bodies are equal in magnitud, opposite in direction and Collinear.

Velocity#

The rate of change of displacement (P.V) is called velocity



$$\underline{v} = \frac{d\underline{r}}{dt} \longrightarrow ①$$

If a particle moves a distance δx in time δt. Then $\frac{\delta x}{\delta t}$ = average velocity

If we take limit when $\delta t \to 0$, then

$$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \text{Velocity}$$

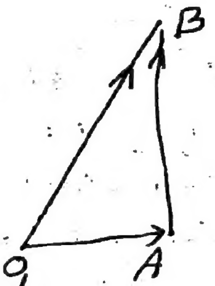
$$\text{ACC} = \underline{a} = \frac{dV}{dt}$$

Vector Addition of Forces#

Forces acting on a particle obey the laws of vector addition. We state it by two methods

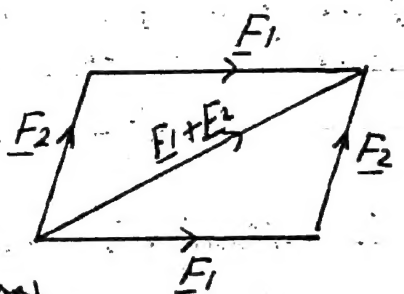
(a) Triangle Law#

$$\vec{OA} + \vec{AB} = \vec{OB}$$



(b) # Parallelogram Law #

If two forces are represented as the adjacent sides of a parallelogram, then their resultant is reprensented by the diagonal of parallelogram. This method is called the parallelogram law.



Types of Vectors#

(a)# Free vector (as usual here)

(b)# Sliding vector (like a force on a rigid body which acts along a line

(c)# Localized vector (Bounded vector) Like a force on a particle which acts upon a point.

Equilibrium Law#

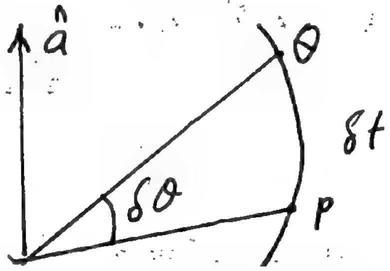
It states that forces are in equilibrium (i-e their mechanical effect is zero) if

(i) they are equal in magnitude
(ii) they are opposite in direction
(iii) they act upon a same line i.e their line of action is same

Th Principal of Superposition of Forces#

The effect of forces acting on a particle or body are mutually idependent i.e the forces act on a particle independently.

Angular Velocity#



If a body is at point P and after a short interval of time δt it reaches at point Q, then the change in angular displacement is $\delta\theta$.

Angular velocity is defined as the rate of change of angular Co-ordinate w.r.t t and is denoted by $\underline{\omega}$

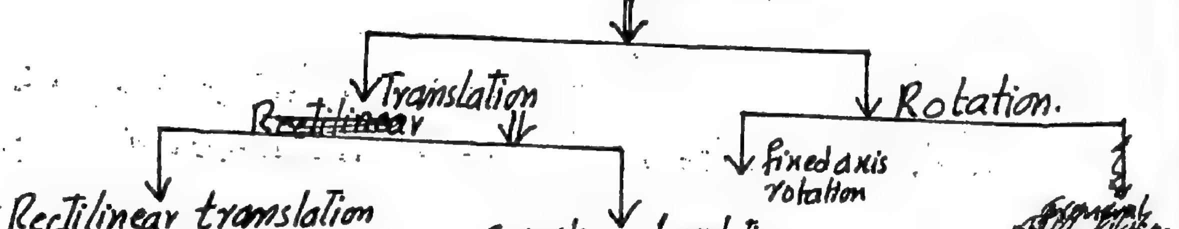
$$\text{i.e}\quad \underline{\omega} = \frac{d\theta}{dt}\hat{a}$$

Its direction is determined by right hand rule

Types of the Plane Motion of a Rigid Body#

The plane motion of a rigid body may be divided as

Plane Motion#



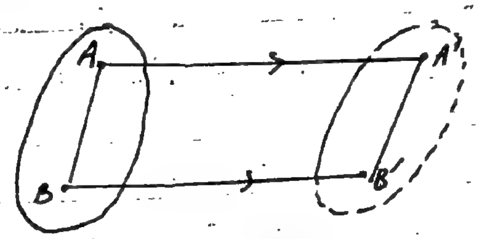
Translation

Any motion in which every line fixed in the body remains parallel to its original position at all times. It is of two types

(a) Rectilinear translation (b) Curvilinear translation.

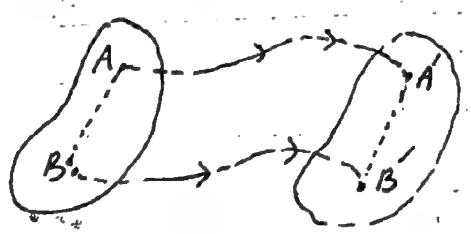
Rectilinear Translation

It is a translation in which all points in the body move in straight lines.



Curvilinear Translation

It is the translation in which all points move on congruent curves. In curvilinear translation there is no ratation of any line in the body



Remarks # It should be noted that in each case of translation the motion of the body is completely specified by the motion of any point since all points have the same motion.

Rotation about Fixed Axis

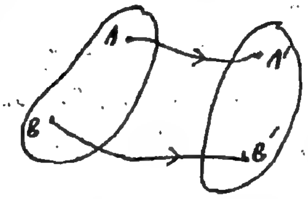
It is the angular motion about the fixed axis. It follows

that all particles move in circular paths about the axis of rotation and all lines in the body (including those that do not pass through the axis) rotate through the same angles in the same time.

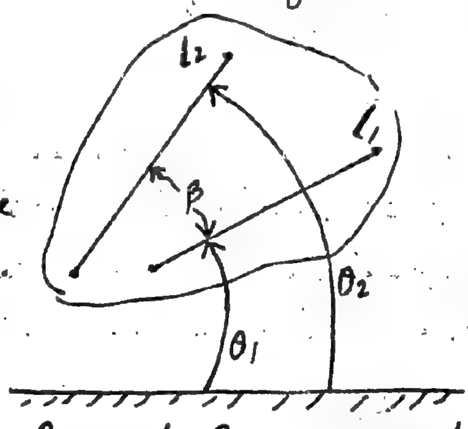
Remarks# It should be noted that in all types of above cited motion, all particles in the body ~~is~~ move in parallel planes. The motion, however, is represented by its projection onto a single plane parallel to the motion and this plane is called plane of the motion. This plane is usually considered as passing through the centre of mass of the body.

General Plane Motion of Rigid Body#

It is a combination of translation and rotation



Problem# Prove that all lines in a rigid body in its plane of motion have the same angular displacement, the same angular velocity and the same angular acceleration



Sol# fig shows a rigid body which undergoes plane motion in the plane of figure. The angular position of any two lines l_1 & l_2 attached to body are specified by θ_1 and θ_2 measured from some fixed reference direction which is convenient. Since the angle β is invariant

Therefore the relation
$$\theta_2 = \theta_1 + \beta$$
gives
$$\dot{\theta}_2 = \dot{\theta}_1 \quad \rightarrow (1)$$
and
$$\ddot{\theta}_2 = \ddot{\theta}_1 \quad \rightarrow (2)$$
or during a finite interval
$$\Delta\theta_2 = \Delta\theta_1$$
Thus all lines in a rigid body in its plane of motion have the same angular displacement, the same angular velocity and the same angular acceleration.

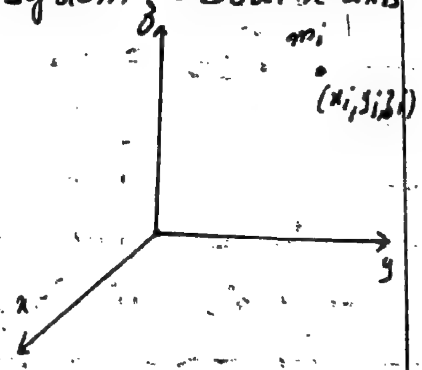
Mass Moment of Inertia About an Axis#

When a rigid body rotates about a fixed axis then moment of inertia (or sec moment) of ith particle m_i is given by
$$m_i r_i^2$$
where m_i is the mass of ith particle and r_i is the distance of ith particle from the axis of rotation.

The moment of inertia of the whole body is given by
$$\sum_i m_i r_i^2$$
where summation is taken over the all the particles of the body.

Moment of Inertia of System about axes#

(1) Moment of inertia of a system about x-axis is
$$\sum_i m_i (y_i^2 + z_i^2)$$



Where (x_i, y_i, z_i) are the Co-ordinates of m_i w.r.t the Co-ordinates system OXYZ.

Here $\sqrt{y_i^2+z_i^2}$ is distance of m_i from x-axis which is calculated by distance of (x_i, y_i, z_i) from point $(x_i, 0, 0)$ on x-axis using distance formula

(2)# Moment of inertia of the system about y-axis is

$$\sum_i m_i(x_i^2+z_i^2)$$

(3) Moment of inertia about z-axis is

$$\sum_i m_i(x_i^2+y_i^2)$$

Products of Inertia w.r.t Axes#

The expressions $\sum m_i y_i z_i$, $\sum m_i z_i x_i$, $\sum m_i x_i y_i$ are called products of inertia of the system about y & z-axis, z & x-axis, x & y-axis

Note# (1) A moment of inertia is always +ve
(2) A product of inertia may be +ve or -ve

Moment of Inertia for Continuous Distribution of Mass#

In case of continuous distribution of mass

I_{xx} = Moment of inertia about x-axis

$$= \int (y^2+z^2)\, dm$$

where dm is the element of mass (mass element) at (x, y, z)

In case of thin uniform rod, density ρ is

and we have

$$dm = \rho\, dx$$

where dx element of differential length.

In case of flat Lamina

$$dm = \rho\, dx\, dy = \rho\, dA$$

ρ is the density at poin (x, y)

In plane polar Co-ordinates.

$$dm = \rho\, r\, d\theta\, dr$$

In case of solid sphere

$$dm = \rho\, dx\, dy\, dz = \rho\, dV.$$

In case of spherical polar Co-ordinates

$$dm = \rho\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Polar Moment of Inertia#

The sum

$$I_O = I_x + I_y$$

is called the polar moment of inertia or the moment of inertia w.r.t origin.

First Moments About the Co-ordinates Planes=

$$M_{yz} = \iiint\limits_{\text{Region of solid}} x\, \rho\, dv \qquad M_{xz} = \iiint\limits_{R} y\, \rho\, dv$$

$$M_{xy} = \iiint\limits_{R} z\, \rho\, dv$$

Centre of Mass

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

where m is total mass

Moments of Inertia (Second Moment)

$$I_x = \iiint_R (y^2+z^2)\,\rho\,dV$$

$$I_y = \iiint_R (x^2+z^2)\,\rho\,dV$$

$$I_z = \iiint_R (x^2+y^2)\,\rho\,dV$$

Moment of Inertia about any line L is

$$I_L = \iiint r^2 \rho\,dV$$

$r(x,y,z)$ = distance from the points (x,y,z) to line L.

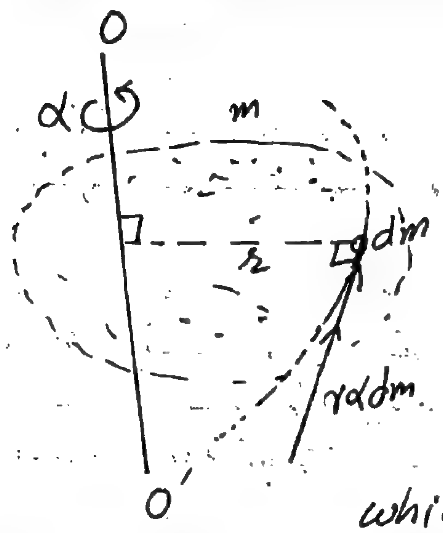
Notations For Moments of Inertia

For plane motion if the motion is about an axis normal to the plane of motion, a single subscript for I is enough to denote the inertia axis. So if a flat lamina has plane motion in xy-plane, then moment of inertia of lamina about the z-axis through o is denoted by I_o

For three dimensional motion, where the components of rotation may may occur about more than one axis, we use double subscripts to preserve notational symmetry with product of inertia terms. Thus, the moments of inertia about the x-y and z-axes are denoted by

I_{xx} , I_{yy} , I_{zz}

Physical Significance of Moment of Inertia



Consider a body of mass rotating about an axis OO' with an ~~(accelerati)~~ angular acceleration α. All particles of the body move in parallel planes which are normal to the rotation axis OO'. Any one of these planes may be considered the plane of motion of body although usually the plane containing the centre of mass is taken as the plane of motion. An element of mass dm has a component of acceleration tangent to its circular path equal to $r\alpha$ ($a = r\alpha$).

By Newton's 2nd Law of motion the resultant tangential force on this element dm is

$$r\alpha\, dm.$$

The moment of this force about the axis OO' is

$$r(r\alpha\, dm) = r^2 \alpha\, dm.$$

and the sum of the moments of these forces for all elements is

$$\int r^2 \alpha\, dm$$

For a rigid body α is same for all radial lines (all radial lines move with sam- angular velocity, acc in a given time) and may

be taken out of the integral sign.

The remaining integral

$$\int r^2 dm$$

is known as the moment of intertia I of of the mass m about the axis OO and is

$$I = \int r^2 dm$$

This integral represents an important of a body and is involved in the force analysis of any body that has rotational acceleration about a given axis.

Now inertia is intrinsic ( ) reluctance or resistance (to accept) which a body offers to accept a change in its state of motion. In case linear motion the mass of a body is the measure of reluctance to the translational acceleration: In case of rotational motion the moment of inertia is a measure of resistance to rotational acceleration of the body.

The moment of inertia of a body is a function of

(i) the mass of body (ii) the distribution of the mass of body (e.g size and shape)
(iii) and the position of axis of rotation.

Radius of Gyration

It is the distance k from the axis of rotation such that if all the mass m of body were concentrated at distance k, the correct moment of inertia would be $k^2 m = I$.

The radius of gyration of a mass m about an

axis for which the moment of inertia is I is

$$k = \sqrt{\frac{I}{m}}$$

Thus k is the measure of distribution of mass of a given body about the axis in question.

Centre of Mass of System of Paricles & Sum of Mass Moments about C.M

Centre of mass of a system of particles $m_1, m_2 - - - - - m_n$ with p.v $\underline{r}_i$ relative to O is defined as.

$$\underline{r}_{c.m} = \frac{\sum m_i \underline{r}_i}{\sum m_i = m}$$

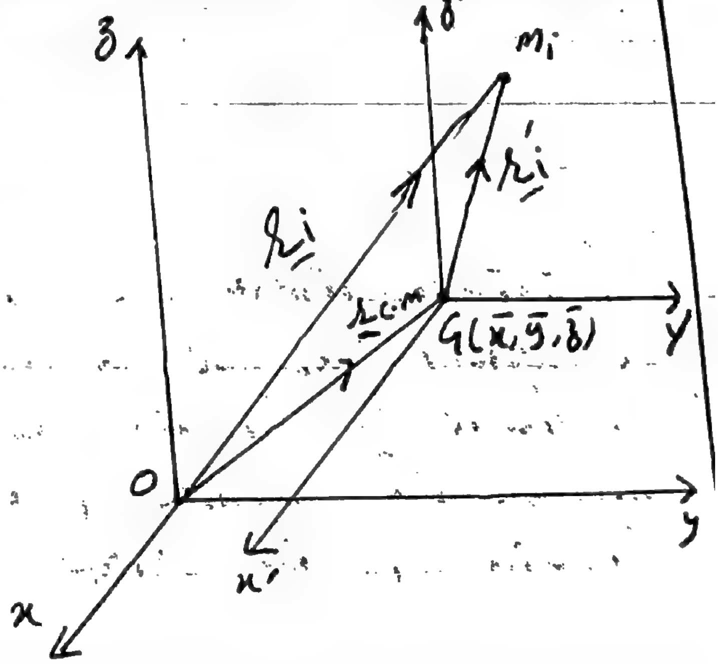
Co-ordinates of c.m are given by

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} \qquad \bar{y} = \frac{\sum m_i y_i}{\sum m}$$

$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

where $\sum m_i \underline{r}_i$ is called sum of mass moments of all the particles.

Let $Gx'y'z'$ be Co-ordinate system with origin at the c.m. Let $\underline{r}_i$ be p.v w.r.t O and $\underline{r}_i'$ be p.V w.r.t G of ith particle m_i



$$\underline{r}_i = \underline{r}_{c.m} + \underline{r}_i'$$

$$\underline{r}_i' = \underline{r}_i - \underline{r}_{c.m}$$

$$\Rightarrow \sum m_i \underline{r}_i' = \sum m_i \underline{r}_i - \underline{r}_{c.m} \sum m_i$$

$$= \sum m_i r_i - \left(\frac{\sum m_i \underline{r}_i}{\sum m_i}\right) \sum m_i$$

$$= \sum m_i \underline{r}_i - \sum m_i \underline{r}_i$$

$$\sum m_i \underline{r}_i' = 0 \qquad \rightarrow (1)$$

$\Rightarrow$ sum of mass moments (about) w.r.t C.m is zero. So we can define C.m of a system as a point w.r.t which the sum of linear moments is zero.

From (1)

$$\sum m_i x_i' = 0 \qquad \sum m_i y_i' = 0$$

$$\sum m_i z_i' = 0$$

We note that x_i' is distance of particles from $y_i' z_i'$ - plane (or from y'-axis, or z'-axis)

Parallel Axis Theorem

OR

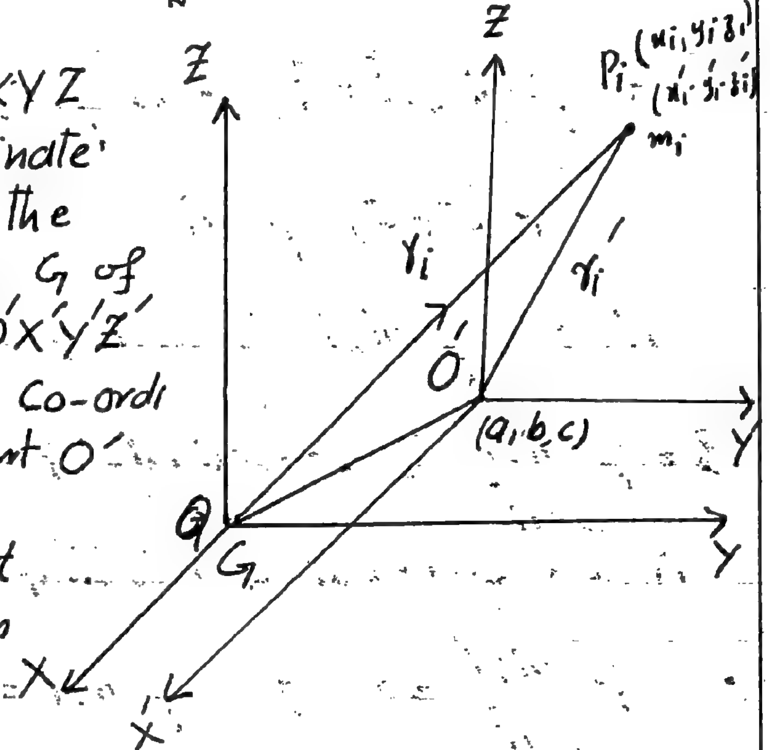
Priciple of Parallel Axes # (Transfer of Axis)

Statement # (1) The moment of inertia of a system about a line L is equal to moment of inertia about a parallel line through the centre of mass plus the moment of inertia of the whole mass centrated at

the centre of mass about line l

(2) The product of inertia of a system about two lines l, m (not passing through the c.m) is equal to the product of inertia about parallel lines (one parallel to l, other parallel to m) plus product of inertia about l, m of the whole mass $M = \Sigma m_i$ placed at the centre of mass.

<u>Proof</u># Let OXYZ be the Co-ordinate system through the centre of mass G of a system. Let $O'X'Y'Z'$ be a parallel co-ordinates through point O' with co-ordinates (a, b, c) w.r.t Co-ordinate system through G.



Let m_i placed at point P_i has Co-ordinates (x_i, y_i, z_i) and (x_i', y_i', z_i')

Then
$$x_i' = x_i - a$$
$$y_i' = y_i - b$$
$$z_i' = z_i - c$$

Now
$$I_{x'x'} = \sum_i m_i (y_i'^2 + z_i'^2)$$
$$= \sum_i m_i \left[(y_i - b)^2 + (z_i - c)^2 \right]$$

$$= \sum_i m_i \left[y_i^2 - 2by_i + b^2 + z_i^2 - 2cz_i + c^2 \right]$$

$$= \sum_i m_i y_i^2 - 2b \sum m_i y_i + \sum_i m_i z_i^2 + (b^2 + c^2) \sum_i m_i$$
$$- 2c \sum_i m_i z_i$$

$$= \sum_i m_i (y_i^2 + z_i^2) - 2b \sum_i m_i y_i - 2c \sum_i m_i z_i$$
$$+ (b^2 + c^2) \sum_i m_i$$

Now $\sum_i m_i y_i$ is sum of moments &

$\frac{\sum_i m_i y_i}{\sum m_i}$ is Co-ordinate of c.m along y-axis through c.m. But c.m is itself origin. Therefore

$$\frac{\sum_i m_i y_i}{\sum m_i} = 0 \Rightarrow \sum_i m_i y_i = 0$$

Similarly $\sum m_i z_i = 0$

Hence

$$I_{x'x'} = \sum_i m_i (y_i^2 + z_i^2) + (b^2 + c^2) M$$

$$= I_{xx} + (b^2 + c^2) M$$

Here $M = \sum_i m_i$

I_{xx} = Moment of intertia about x-axis through G

$I_{x'x'}$ = Moment of inertia about a line parallel to x-axis

$(b^2 + c^2) M$ = Moment of intertia of totat mass

~~abou~~ at c.m about line x'-axis.

Similarly

$$I_{y'y'} = I_{yy} + M(a^2+c^2)$$

$$I_{z'z'} = I_{zz} + M(a^2+b^2)$$

Product of inertia about x'-axis, y'-axis

$$= I_{x'y'} = \sum_i m_i x_i' y_i'$$

$$= \sum_i m_i (x_i - a)(y_i - b)$$

$$= \sum_i m_i x_i y_i - a \sum_i m_i y_i - b \sum m_i x_i + \sum_i m_i ab$$

$$= \sum_i m_i x_i y_i + 0 + 0 + \sum_i m_i ab$$

$$= I_{xy} + Mab$$

$\Rightarrow$ Product of interlia about two lines x', y'

= Product of inertia about parall lines through c.m

+ Product of inertia of whole mass M at c.m about lines x', y'

Hence the parallel axes theorem is proved.

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Perpendicular Axes Theorem#

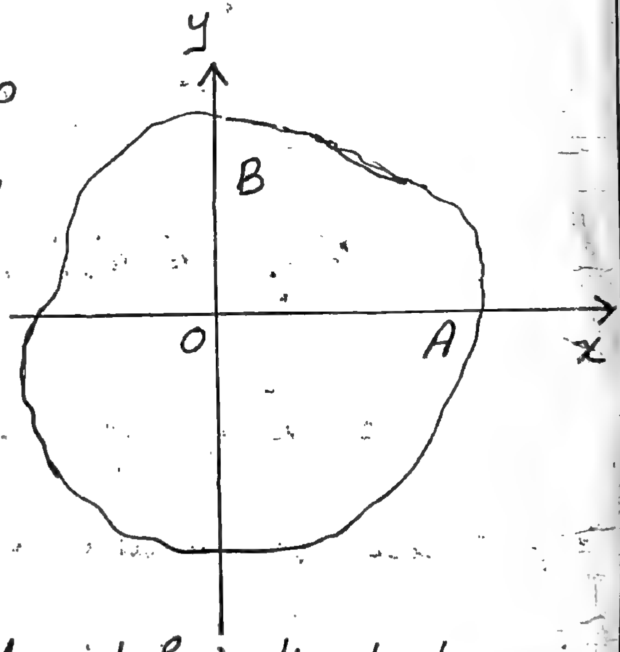
OR

Lamina Theorem#

Statement# If A and B are the moments of inertia of a (flat) Lamina about two perpedicular axes in its plane, then its moment of inertia C about the line through the point of intersection of these axes and perpendicular to these is given by

$$C = A + B$$

Proof# Taking the two perpendicular lines in lamia as x-axis and y-axis respectively and the line through their point of intersection, perpendicular to the lamia as z-axis, we have



$A = \int (y^2 + z^2)\,dm$ = Moment of inertia about x-axis

$= \int y^2\,dm$ as $z = 0$

$B = \int (z^2 + x^2)\,dm$ = Moment of inertia about y-axis

$= \int x^2\,dm$ $z = 0$

Moment of inertia about z-axis

$$= C = \int (x^2+y^2)\,dm$$

$$= \int x^2 dm + \int y^2 dm$$

$$C = A + B \quad \text{proved} \longrightarrow (1)$$

If K_1, k_2, k_3 are radii of gyration about x-axis, y-axis and z-axis respectively. Then

$$A = Mk_1^2 \quad B = Mk_2^2 \quad C = Mk_3^2$$

Then from (1) we have

$$Mk_3^2 = Mk_1^2 + Mk_2^2$$

$$\Rightarrow k_3^2 = k_1^2 + k_2^2$$

Remarks# 1)# Lamina theorem is useful to calculate moment of inertia of any plane (Lamina, ring, triangular framework etc) about certain axes.

2)# This theorem can not be applied to three dimensial rigid bodies.

3)# The three axes in theorem must be mutually perpendicular and concurrent, although none of them need pass through the centre of mass of body

4)# The parallel axes theorem given before is more general and can be applied to three dimensional rigid bodies.

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Routh's Rule

The following rule given by Dr. Routh is useful for remembering the moments of inertia of certain symmetrical solid bodies about their principal axes and symmetry axes.

M·I about an axis of symmetry

$$= \text{Mass} \times \left[\frac{\text{The sum of squares of perp. semi-axes}}{3 \text{ or } 4 \text{ or } 5}\right]$$

$\Downarrow$

Value of k^2 (square of radius of gyration)

The denominator is 3 or 4 or 5 according as the body is rectangular (including the rod and a square) or elliptical (including a circle) or ellipsoidal (including a sphere)

Moment of Inertia of a Uniform Rod

Question # (a) # Find the moment of inertia of a Thin uniform rod about axis (i) through the centre (c·m) and perp. to rod (ii) through the centre and inclined at angle θ to rod. (iii) through one end of rod and perp to rod.

(b) # Find the moment of inertia of a uniform rod about an axis (i) through the centre (ii) through centre and inclined at angle θ to the rod (iii) through one end of rod and perp to the rod.

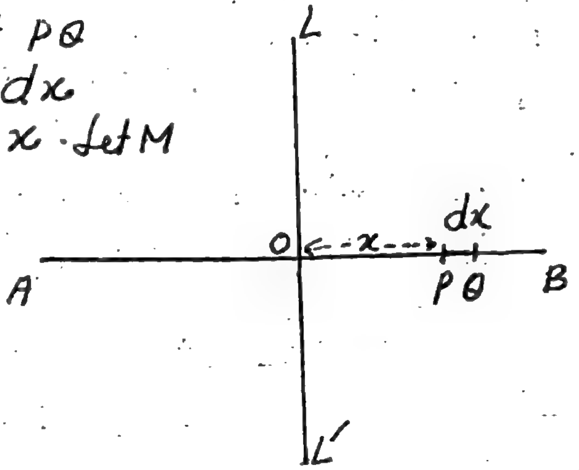
Sol # (a) Thin Uniform Rod #

Let $AB = 2a$ be a thin rod of uniform

density ρ so that mass per unit length is same through out the rod

(i) Suppose we require the moment of inertia of the rod about the axis LOL' which bisects the rod perpendicularly at O.

Consider an element PQ of differential length dx at a distance $OP = x$. Let M be total mass of the rod



Differential mass element $= dm = \rho\, dx$

But $\rho = \frac{\text{Mass}}{\text{Length}} = \frac{M}{2a}$

$$\Rightarrow dm = \left(\frac{M}{2a}\right) dx$$

Moment of inertia of mass dm about LOL' is

$$dI_{LL'} = dI_{yy} = x^2\, dm$$

Moment of inertia of rod a

$$= I_{LL'} = I_{yy} = \int_{-a}^{a} x^2\, dm$$

$$= \int_{-a}^{a} \left(\frac{M}{2a}\right) \cdot x^2\, dx$$

$$= \frac{M}{2a} \left|\frac{x^3}{3}\right|_{-a}^{a} = \frac{M}{2a}\left[\frac{a^3}{3} + \frac{a^3}{3}\right]$$

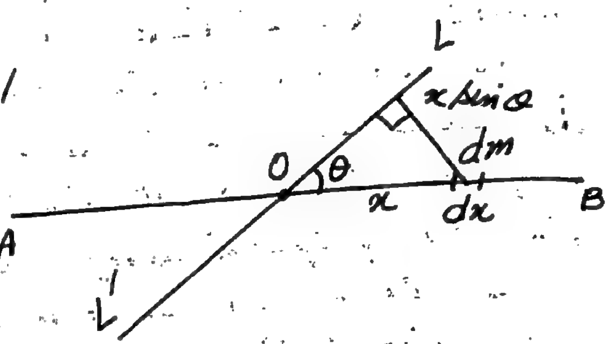
$$\boxed{I_{yy} = \frac{1}{3} Ma^2} \Rightarrow \text{Result}$$

(ii) # Suppose we want the moment of inertia about an axis LOL' through mid point and

as shown:

Consider a differential length element dx at distance x from O.

Perp distance of dx from $LOL' = x \sin\theta$



Differential Mass $= dm = \rho\, dx$

Differential momoment of inertia of dm about LOL'

$$= dI_{LL'} = x^2 \sin^2\theta\, dm$$

$$= \rho \sin^2\theta\, x^2 dx$$

Moment of inertia of rod about LOL' is

$$I_{LL'} = \int_{-a}^{a} \rho \sin^2\theta\, x^2 dx$$

$$= \rho \sin^2\theta \left| \frac{x^3}{3} \right|_{-a}^{a}$$

$$= \frac{M}{2a} \sin^2\theta \left[\frac{a^3}{3} + \frac{a^3}{3} \right]$$

$$= \frac{M}{2a} \sin^2\theta \cdot \frac{2a^3}{3}$$

$$= M\frac{a^2}{3} \sin^2\theta$$

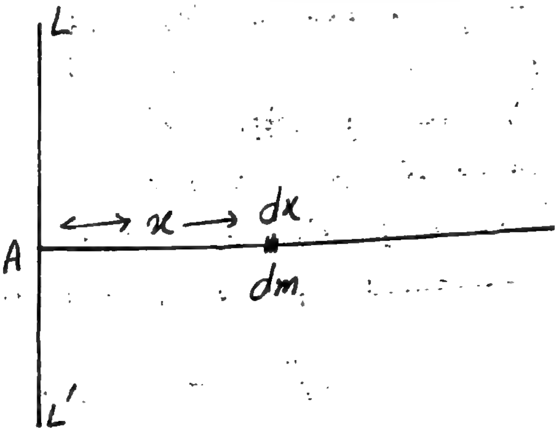
$$\boxed{I_{LL'} = M\frac{a^2}{3} \sin^2\theta} \Longrightarrow \text{Result}$$

<u>Deductions</u>: Moment of inertia of thin uniform rod about an axis through centre and inclined at angles of 30°, 45°, 60°, 90° is given

$$\frac{Ma^2}{12}, \quad \frac{Ma^2}{6} \quad \frac{Ma^2}{4} \quad \frac{Ma^2}{3}$$

We note that moment of inertia about an axis which has greater inclination to rod is greater.

(iii) # Moment of inertia about an axis passing through end A is given by



$$I_{LL'} = \int_0^{2a} x^2 dm$$

$$= \int_0^{2a} x^2 \cdot \frac{M}{2a} dx$$

$$= \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = \frac{4}{3}Ma^2.$$

OR By parallel axes theorem

Momement of inertia about an axis through one ⊥ar to rod

= Moment about about Centroidal ⊥ar axis
+ Moment of whole mass at centre about axis through end

$$= \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2$$

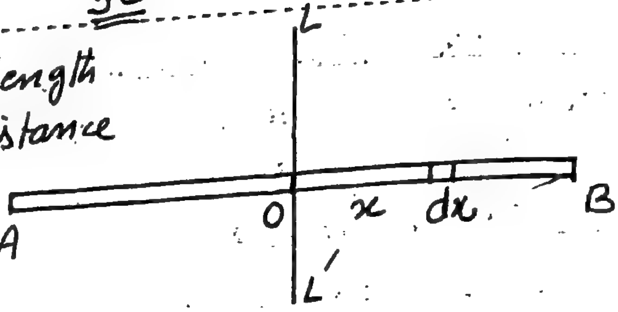
(b)

Uniform Rod# AB

Consider a uniform rod of length $2a$ and of Cross-Sectional area α. O is middle point of rod.

Uniform density ρ of rod is

$$\rho = \frac{Mass}{Volume} = \frac{M}{2a\alpha}$$ [Volume = Area of Cross-section × length



(i). Consider differential length element dx at distance x from the centre of rod.

Volume of differential element $= \alpha\, dx$

Mass of the element $= \rho\alpha\, dx = dm$.

Moment of mass dm about LOL' is

$$dI_{LL'} = x^2 \rho\alpha\, dx$$

Moment of inertia of rod is

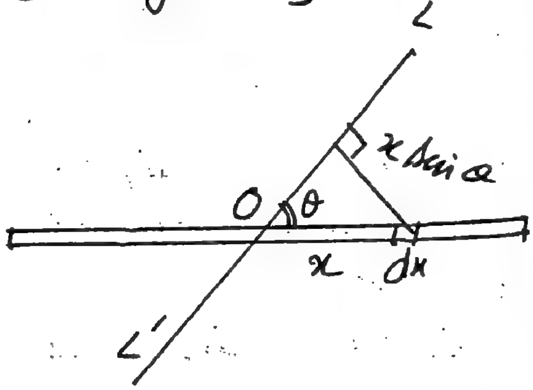
$$I_{LL'} = \rho\alpha \int_{-a}^{+a} x^2\, dx$$

$$= \left(\frac{M}{2a\alpha}\right)\alpha \left[x^3\right]_{-a}^{a}$$

$$= \frac{M}{2a}\left[\frac{2a^3}{3}\right]$$

$$\boxed{I_{LL'} = \frac{Ma^2}{3}} \longrightarrow \text{Result}$$

(ii) Moment of inertia about centroidal axis inclined at an angle θ is given by



$$dm = \rho\alpha\, dx$$

$$dI_{LL'} = x^2 \sin^2\theta\, dm$$

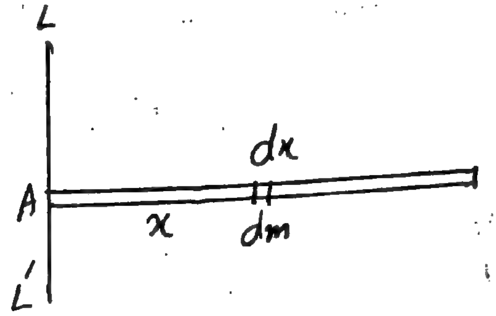
$$= \rho\alpha \sin^2\theta \cdot x^2 dx$$

$$I_{LL'} = \int_{-a}^{a} \rho\alpha \sin^2\theta \cdot x^2 dx$$

$$= \left(\frac{M}{2a\alpha}\right)\alpha \cdot \sin^2\theta \int_{-a}^{a} x^2 dx$$

$$I_{LL'} = \frac{Ma^2}{3}\sin^2\theta$$

(iii) Moment of inertia about an axis LL' passing through end A is given by



$$dm = \rho\alpha\, dx$$

$$dI_{LL'} = x^2 dm$$

$$= (\rho\alpha\, dm)dx$$

$$= \frac{M}{2a\alpha}\cdot\alpha\cdot x^2 \cdot dx$$

$$= \frac{M}{2a}\, x^2\, dx$$

$$I_{LL'} = \frac{M}{2a}\int_0^a x^2 dx = \frac{M}{2a}\cdot\frac{8a^3}{3} = \frac{4Ma^2}{3}$$

$$\boxed{I_{LL'} = \frac{4Ma^2}{3}} \Longrightarrow \text{Result}$$

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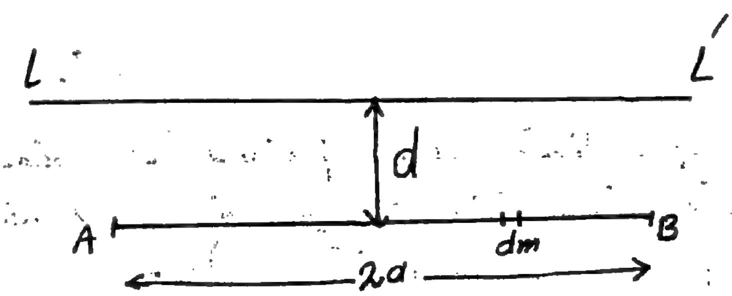
Moment of Inertia of Uniform Thin Rod About an axis Parallel to Rod

Question # Find the moment of inertia of a uniform thin rod of Mass M and length $2a$ about an axis parallel to the rod at distance (d) from it.

Sol #



Consider rod $AB = 2a$ of uniform density ρ. Consider a differential mass element dm.

Moment of dm about $LL' = dI_{LL'} = d^2 dm$

Moment of inertia of rod about LL' is

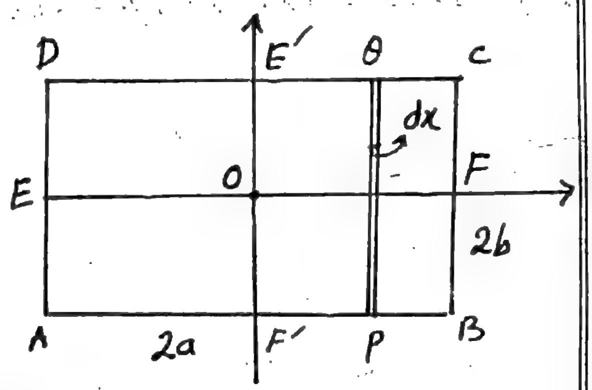
$$I_{LL'} = \int d^2 dm = d^2 \int dm$$

$$= d^2 M \qquad \int dm = M$$

Moment of Inertia of Uniform Rectangular Lamina

Question # Find the moment of Inertia of a Uniform rectangular Lamina of sides $2a$, $2b$ about
(i) Centroidal axes parallel to sides
(ii) an axis passing through centre and perp to the lamina (iii) Product of inertia about centroidal axes parallel to sides.

Sol # Let ABCD be a uniform rectangular Lamina of density ρ and mass M. If E, F are mid points of sides AD, BC and E', F' are mid points of sides CD, AB, then EF, $E'F'$ are centroidal axes because Lamina is uniform.



Moment About EF

Consider a thin strip $P\theta$ of width dx parallel to side AD. Thin strip is like thin rod of uniform density and axis EF passes through its centre.

Moment of inertia of strip is

$$dI_{xx} = dI_{EF} = \frac{1}{3} b^2 dm.$$

(This moment is calculated by formula of moment of inertia of a thin uniform rod about an axis passing through middle point of the rod and in the plane of rod)

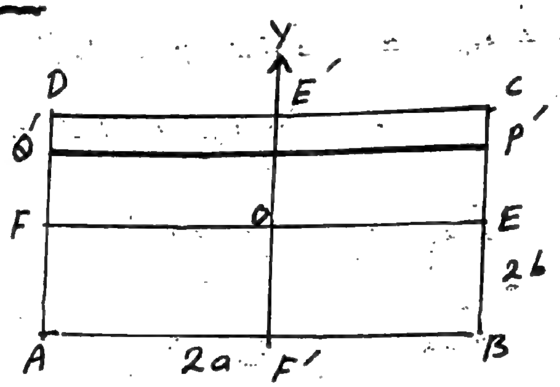
Moment of inertia of the whole lamina about EF is

$$I_{EF} = I_{xx} = \frac{1}{3}b^2\int dm$$

$$I_{xx} = \frac{1}{3}b^2M \quad \rightarrow (1) \qquad \because \int dm = M \text{ whole mass of Lamina}$$

(i) # <u>Moment About $E'F'$</u> #

Consider a thin strip $P'Q'$ of width dy parallel to side AB.



Then $P'Q'$ is a thin uniform rod of length $2a$ and $E'F'$ is an axis at middle point of it. So

Moment of inertia of $P'Q'$ about $E'F'$ is

$$dI_{E'F'} = \frac{1}{3}a^2 dm \qquad \text{(By thin uniform rod method)}$$

⇒ Moment of the whole lamina about $E'F'$ is

$$I_{yy} = I_{E'F'} = \frac{1}{3}a^2\int dm$$

$$I_{yy} = \frac{1}{3}a^2M \quad \rightarrow (2)$$

(ii) # <u>Moment of Inertia about an axis through centre and perp. to Lamina</u> #

Now by Theorem of perpendicular axes the moment of inertia about an axis through centre and perpendicular to lamina is

$$\frac{1}{3}b^2M + \frac{1}{3}a^2M = \frac{1}{3}(a^2+b^2)M$$

Moment About Sides of Lamina

By using parallel axes theorem we can find moment of inertia about sides as.

Moment of inertia about side AB
= Momoment of inertia about side CD
= Moment about EF + Moment of whole mass M at O about AB or CD

$$I_{AB} = I_{EF} + Mb^2$$

$$= \frac{1}{3}Mb^2 + Mb^2 = \frac{4}{3}Mb^2$$

$$\boxed{I_{AB} = \frac{4}{3}Mb^2} \Longrightarrow \text{Result}$$

$$I_{CD} = \frac{4}{3}Mb^2$$

Similarly

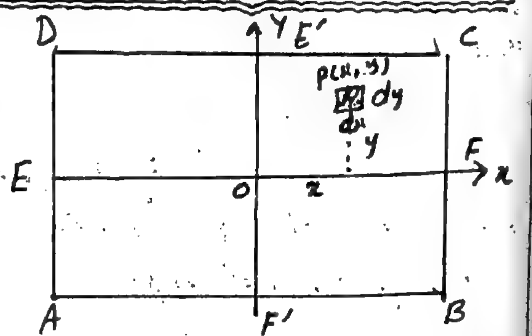
$$I_{AD} = I_{BC} = I_{E'F'} + Ma^2$$

$$= \frac{1}{3}Ma^2 + Ma^2$$

$$\boxed{I_{AD} = I_{BC} = \frac{4}{3}Ma^2} \Longrightarrow \text{Result}$$

(iii) # Product of Moment of Inertia About Centroidal axes

Consider a differential area element dxdy at point P(x, y). Mass of this element is



$$dm = \rho\, dx\, dy$$

Product of inertia of mass element dm about x-axis & y-axis is

$$dI_{xy} = xy\,dm$$

$$= \rho\, x y\, dx\, dy$$

So product of inertia of the whole lamina about x-axis and y-axis is

$$I_{xy} = \int_{-b}^{b}\int_{-a}^{a} \rho\, x y\, dx\, dy$$

$$= \rho \int_{-b}^{b} \left(\int_{-a}^{a} x\, dx \right) y\, dy$$

$$= \rho \int_{-b}^{b} \left[\frac{x^2}{2} \right]_{-a}^{a} y\, dy$$

$$= \rho \int_{-b}^{b} \left(\frac{a^2}{2} - \frac{a^2}{2} \right) y\, dy$$

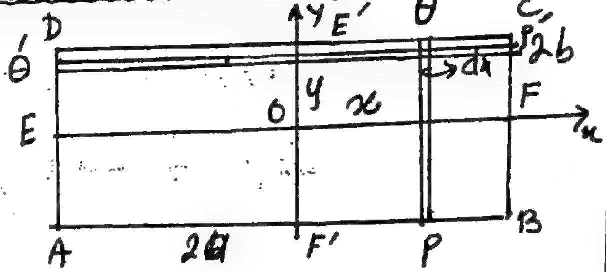
$$= \rho \int_{-b}^{b} 0\, y\, dy = 0$$

$\Rightarrow$ Centroidal axes are infact principal axes for lamina.

II- Method

(i) Moment About Centroidal Axes

For moment of inertia about $E'F'$ Consider a strip of width dx parallel to



Side AD.

Length of the strip $= 2b$

Area of the strip $= 2b\,dx$

If ρ is density of Lamina, then mass of the strip $= dm = \rho \cdot 2b\,dx$

Moment of inertia of dm about $E'F'$ is

$$dI_{E'F'} = 2b\rho\, x^2 dx$$

$$\Rightarrow I_{E'F'} = 2b\rho \int_{-a}^{a} x^2 dx$$

$$= 2b\rho \left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$= 2b\rho \left[\frac{2a^3}{3}\right]$$

$$= \frac{4\rho b a^3}{3}$$

Area of lamina $= 2a \times 2b = 4ab$

total Mass $= M$

$$\rho = \frac{Mass}{area} = \frac{M}{4ab}$$

$$\Rightarrow I_{E'F'} = \frac{4}{3} b a^3 \cdot \frac{M}{4ab} = \frac{1}{3} Ma^2$$

Similarly considering a strip of width dy parallel to side AB at distance y from O. We have

Length of the strip $= 2a$

Area of strip $= 2a\,dy$

$dm = 2a\rho\,dy$

$$dI_{EF} = y^2 \cdot 2a\rho\,dy$$

$$I_{EF} = 2a\rho \int_{-b}^{b} y^2 dy = 2a \cdot \frac{M}{4ab} \cdot \frac{2b^3}{3} = \frac{1}{3} Mb^2$$

OR

Consider a small area element $dxdy$ at point (x,y) of Lamina. Then

$$dm = \rho\, dxdy$$

$$I_{xx} = \iint dI_{xx} = \iint dI_{EF} = \iint y^2\, dm$$

$$= \rho \int_{-a}^{a}\int_{-b}^{b} y^2\, dy\, dx$$

$$= \frac{M}{4ab}\int_{-a}^{a}\left[\frac{y^3}{3}\right]_{-b}^{b} dx$$

$$= \frac{M}{4ab}\int_{-a}^{a}\frac{2b^2}{3}\, dx$$

$$= \frac{M}{4ab}\cdot\frac{2b^2}{3}\int_{-a}^{a} dx$$

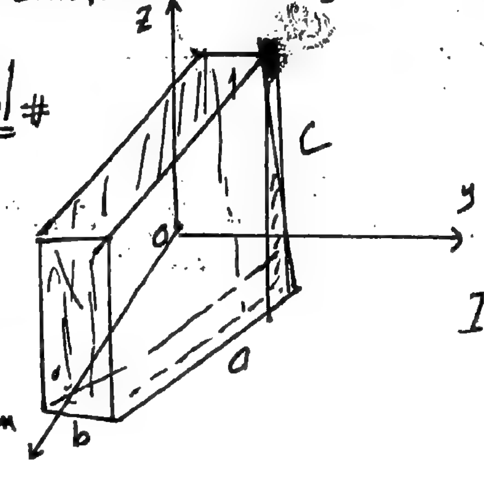
$$= \frac{M}{4ab}\cdot\frac{2b^2}{3}\cdot 2a = \frac{1}{3}Mb^2$$

Similarly $I_{yy} = I_{E'F'} = \frac{1}{3}Ma^2$

Uniform Rectangular Block

Question# Find I_x, I_y, I_z for the rectangular solid of constant density ρ as shown

Sol#



The origin lies at the centre of block

Here $dm = \rho\, dxdydz$

$$I_x = \iiint (y^2+z^2)\, \rho\, dxdydz$$

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2+z^2)\,\rho\, dx\, dy\, dz$$

Since the integrand is an even function.
therefore

$$I_x = 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2+z^2)\,\rho\, dx\, dy\, dz$$

$$= 8\rho \int_0^{c/2}\int_0^{b/2} (y^2+z^2)\, |x|_0^{a/2}\, dy\, dz$$

$$= 8\rho\cdot\frac{a}{2}\int_0^{c/2}\left[\frac{y^3}{3}+z^2y\right]_{y=0}^{y=b/2} dz$$

$$= 4\rho a\int_0^{c/2}\left(\frac{b^3}{24}+\frac{z^2 b}{2}\right) dz$$

$$= 4a\rho\left(\frac{b^3c}{48}+\frac{c^3b}{48}\right) = \frac{abc\,\rho}{12}(b^2+c^2)$$

Now Volume of block $= abc$

Total Mass $= M$

$$\rho = \frac{M}{abc}$$

$$\Rightarrow I_x = \frac{abc}{12}\cdot\frac{M}{abc}(b^2+c^2)$$

$$= \frac{M}{12}(b^2+c^2)$$

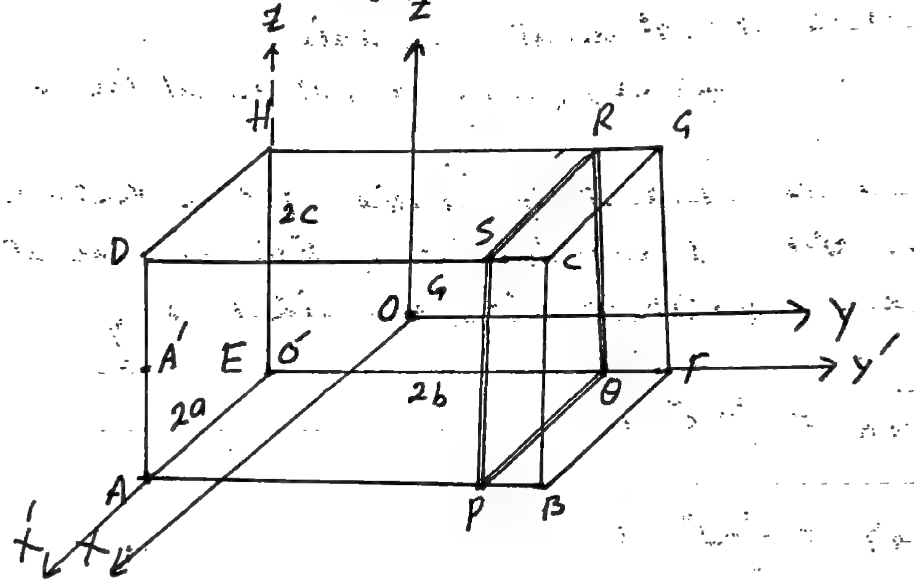
Similarly

$$I_y = \frac{M}{12}(a^2+c^2) \;\&\; I_z = \frac{M}{12}(a^2+b^2)$$

M.I of Homogeneous Rectangular Parallelopiped

Question # Calculate the moment of inertia of homogeneous (uniform) rectangular parallelopiped about centroidal axes, about edges, about axes through centre of faces and perp to centroidal axes. Also find the product of inertia about centroidal axes and about edges.

Sol #



Consider a parallelopiped of edges $2a$, $2b$, $2c$ with centroid (c-m) G. Consider axes through G parallel to respective edges. Let ρ be the uniform density of the parallelopiped.

About Centroidal Axes

To find the moment of inertia about centroidal y-axis, consider a small rectangular element parallel to face ADHE and perpendicular to y-axis. Let dy be its very small thickness as compared the whole.

dimension of parallelopid.

Dimensions of rectangular element PQRS are $2a$, $2c$

Area of element $= 2a \times 2c = 4ac$

Now ρdy = Mass per unit volume × thickness

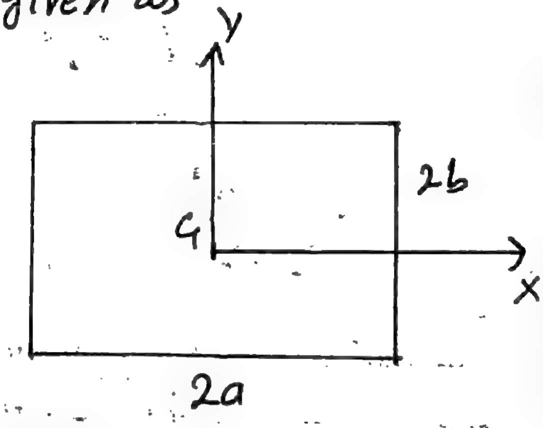
= Mass per unit area → (1)

When thickness t of flat plate is very small, then mass moment and area moment are related as

Mass moment about any axis

$= (\rho t)$ (Area moment about the same axis)

Mass moment of inertia for a rectangular Lamin of length $2a$ & width $2b$ about centroidal axes parall to sids are given as



$$I_{GX} = \frac{1}{3} a^2 M$$

$$I_{GY} = \frac{1}{3} b^2 M$$

and Area moments are

$$I_{GX} = \frac{1}{3} a^2 A$$ where A is total area

$$I_{GY} = \frac{1}{3} b^2 A$$ where A is total area

Polar Moment about an axis through G ⊥ar to Lamina is

$$I_{GX} + I_{GY}$$

Moment of rectangular strip about y-axis $= dI_{yy}$ = Sum of moments about centroiadal

axes parallel to sides

$$dI_{yy} = \left(\frac{1}{3}a^2 A\right)\rho\, dy + \left(\frac{1}{3}c^2 A\right)\rho\, dy$$

$$= \frac{1}{3}\rho\,(a^2+c^2)\,dy \cdot A$$

$$= \frac{1}{3}\frac{M}{8abc}(a^2+c^2)\,dy \cdot 4ac$$

$$= \frac{M}{6b}(a^2+c^2)\,dy$$

<u>OR</u>

Volume of rectangular strip $= 4ac\,dy$

Mass of the strip $= dm = \rho\, 4ac\, dy$

$$= \frac{M}{8abc} \cdot 4ac\,dy$$

$$= \frac{M}{2b}\,dy$$

Polar Moment of strip about axis (Y-axis) through its centre and perp to its plane is

$$dI_{yy} = \frac{1}{3}dm\,(a^2+c^2)$$

$$= \frac{1}{3}\cdot\frac{M}{2b}(a^2+c^2)\,dy$$

$$= \frac{M}{6b}(a^2+c^2)\,dy$$

Momoment of parallelopiped about Y-axis

$$I_{YY} = \int dI_{yy} = \frac{M}{6b}\int_{-b}^{b}(a^2+c^2)\,dy$$

$$I_{yy} = \frac{M}{6b}(a^2+c^2)\,|y|_{-b}^{b}$$

$$= \frac{M}{6b}(a^2+c^2)\cdot 2b$$

$$I_{yy} = \frac{1}{3}M(a^2+c^2)$$

Similarly

$$I_{xx} = \frac{1}{3}M(b^2+c^2)$$

$$I_{zz} = \frac{1}{3}M(a^2+b^2)$$

About Edges

About Edge # AD

Taking G, origin

Co-ordinates point A' are $(a, -b, 0)$

AD is parallel to z-axis and distance (d) between them is

$$d = A'G = \sqrt{a^2+b^2}$$

Moment of Inertia about AD by parallel axis theorem is

$$I_{AD} = I_{zz} + \text{Moment of the whole mass } M \text{ at } G \text{ about } AD$$

$$= \frac{1}{3}M(a^2+b^2) + Md^2$$

$$= \frac{1}{3}M(a^2+b^2) + M(a^2+b^2)$$

$$= \frac{4}{3}M(a^2+b^2)$$

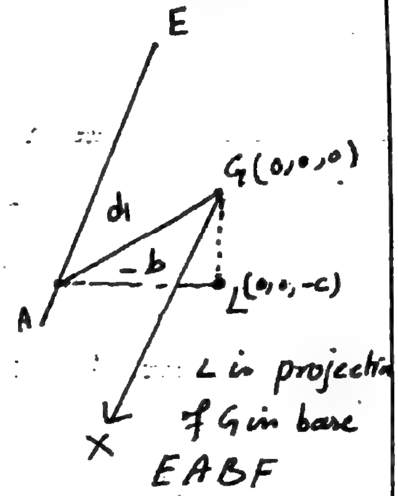
Similarly about edge EH is

$$I_{EH} = \frac{4}{3}M(a^2+b^2) \quad \text{By symmetry}$$

About Edge EA

EA is parallel to X-axis and its distance d_1 from x-axis is calculated as

$$d_1 = \sqrt{b^2+c^2}$$



Moment About EA is

$$I_{EA} = I_{xx} + Md_1^2$$

$$= \frac{1}{3}M(b^2+c^2) + M(b^2+c^2)$$

$$= \frac{4}{3}(b^2+c^2)M$$

Similarly we can calculate moments about other edges

$$I_{EF} = \frac{4}{3}(a^2+c^2)M$$

By Routh's Rule

Since the centroidal axes through G are symmetry axes, therefore we can apply Routh's Rule to memberize the results as

$$I_{xx} = \frac{\text{Mass} \times (\text{Sum of squares of per-semi axes})}{3}$$

$$= \frac{1}{3}M(b^2+c^2)$$

Similarly

$$I_{yy} = \frac{1}{3}M(a^2+c^2)$$

$$I_{zz} = \frac{1}{3}M(a^2+b^2)$$

IInd Method (Direct Method)

$$I_{xx} = \int_V \rho(y^2+z^2)\,dV$$

$$= \rho \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(y^2+z^2)\,dx\,dy\,dz \quad \left[\because \rho \text{ is Constant}\right.$$

$$= \rho\, |x|_{-a}^{a} \int_{-c}^{c}\int_{-b}^{b}(y^2+z^2)\,dy\,dz$$

$$= \rho(2a)\int_{-c}^{c}\left|\frac{y^3}{3}+z^2y\right|_{-b}^{b} dz$$

$$= 2a\rho\int_{-c}^{c}\left(\frac{b^3}{3}+z^2b+\frac{b^3}{3}+z^2b\right)dz$$

$$= 2a\rho\int_{-c}^{c}\left(\frac{2b^3}{3}+2bz^2\right)dz$$

$$= 2a\rho\left|\frac{2b^3}{3}z+\frac{2b}{3}z^3\right|_{-c}^{c}$$

$$= 2a\rho\left[\frac{2b^3}{3}c+\frac{2b}{3}c^3+\frac{2b^3}{3}c+\frac{2b}{3}c^3\right]$$

$$= 2a\rho\left[\frac{4b^3c}{3}+\frac{4b}{3}c^3\right]$$

$$= 2a\rho\cdot\frac{4}{3}bc(b^2+c^2)$$

$$= \rho\,\frac{8abc}{3}(b^2+c^2)$$

putting $\rho = \frac{M}{8abc}$

$$I_{xx} = \frac{M}{8abc} \cdot \frac{8abc}{3}(b^2+c^2)$$

$$= \frac{1}{3}M(b^2+c^2)$$

Similarly

$$I_{yy} = \frac{1}{3}M(a^2+c^2)$$

$$I_{zz} = \frac{1}{3}M(a^2+b^2)$$

About Edges#

Considering O' as origin we can find moment about edges.

$$I_{EA} = I_{O'x'} = \int_V \rho(y^2+z^2)\,dV$$

$$= \rho \int_0^{2c}\int_0^{2b}\int_0^{2a} (y^2+z^2)\,dx\,dy\,dz$$

$$= \rho \left|x\right|_0^{2a} \int_0^{2c}\int_0^{2b} (y^2+z^2)\,dy\,dz$$

$$= 2\rho a \int_0^{2c}\int_0^{2b} (y^2+z^2)\,dy\,dz$$

$$= 2\rho a \int_0^{2c} \left|\frac{y^3}{3} + z^2 y\right|_0^{2b} dz$$

$$= 2\rho a \int_0^{2c} \left(\frac{8b^3}{3} + 2z^2 b\right) dz$$

$$= 2\rho a \left|\frac{8b^3}{3}z + \frac{2z^3}{3}b\right|_0^{2c}$$

$$= \rho\, a \cdot 2 \left| \frac{8b^3 c \cdot 2}{3} + \frac{16 c^3 b}{3} \right|$$

$$= \rho\, \frac{2a\,bc}{3} \,(b^2 + c^2) \cdot 16$$

$$= \frac{32}{3}\, \rho\, abc\,(b^2 + c^2)$$

but $\rho = \frac{Mass}{Volume} = \frac{M}{2a \cdot 2b \cdot 2c} = \frac{M}{8abc}$

$$I_{ox'} = \frac{32}{3} \cdot \frac{M}{8abc} \cdot abc\,(b^2 + c^2)$$

$$= \frac{4}{3} M\,(b^2 + c^2)$$

Similarly

$$I_{oy'} = \frac{4}{3} M\,(a^2 + c^2)$$

and

$$I_{oz'} = \frac{4}{3} M\,(a^2 + b^2)$$

Product of Inertia

(a) **About Edges** #

Considering O' as origin and axes along edges products of inertia about edges are calculated as

$$I_{x'y'} = \int \rho\, xy\, dv$$

Here $dv = dx\,dy\,dz$

$$= \rho \int_0^{2a} x\,dx \int_0^{2b} y\,dy \cdot \int_0^{2c} dz$$

$$= \rho \left|\frac{x^2}{2}\right|_0^{2a} \cdot \left|\frac{y^2}{2}\right|_0^{2b} \cdot |z|_0^{2c}$$

$$= \rho \left(\frac{4a^2}{2}\right) \cdot \left(\frac{4b^2}{2}\right) \cdot 2c$$

$$= \rho \cdot \frac{32a^2b^2c}{4}$$

$$= \rho \cdot 8a^2b^2c$$

$$= \frac{M}{8abc} \cdot 8a^2b^2c = Mab$$

Similarly $I_{x'z'} = Mac$

$I_{y'z'} = Mbc$

About Centroidal Axes#

$$I_{xy} = \rho \int_V xy \, dv$$

$$= \rho \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} xy \, dz\,dy\,dx$$

$\because$ the integrand is an even functi in z but is an odd function in x & y. so

$$I_{xy} = \rho \int_{-c}^{c} 0 \, dz = 0$$

Similarly

$$I_{yz} = I_{zx} = 0$$

Deduction for Uniform Cube of Edge 2a#

Moments of inertia for paralledopiped about Centroidal axes are

$$I_{xx} = \frac{1}{3}M(b^2+c^2)$$

$$I_{yy} = \frac{1}{3}M(a^2+c^2)$$

$$I_{zz} = \frac{1}{3}M(a^2+b^2)$$

Moments of inertia for cube about centroidal axes are

$$I_{xx} = \frac{1}{3}M(a^2+a^2) = \frac{2}{3}Ma^2$$

$$I_{yy} = \frac{1}{3}M(a^2+a^2) = \frac{2}{3}Ma^2$$

$$I_{zz} = \frac{1}{3}M(a^2+a^2) = \frac{2}{3}Ma^2$$

Moment of inertia for parallelopiped about edge AD is

$$I_{AD} = \frac{4}{3}M(a^2+b^2)$$

For cube

$$I_{AD} = \frac{4}{3}M(a^2+a^2)$$

$$= \frac{8}{3}Ma^2$$

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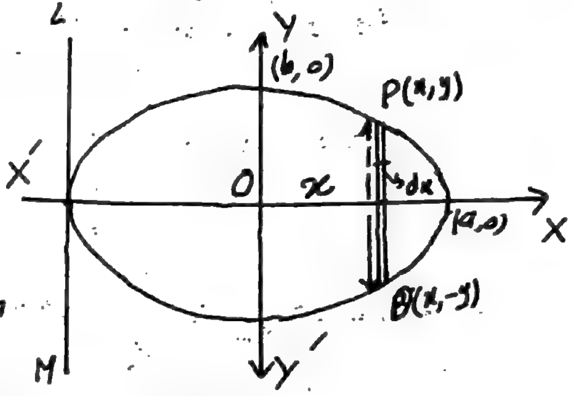
M.I of Uniform Elliptic Lamina or Disc

Question# Find the moment of Inertia of a uniform elliptic Lamina of major and minor axes $2a$, $2b$ about
(i) Centroidal axes (ii) an axis through intersection of centroidal axes and perp to the plane of Lamina
(iii) tangents at the ends of centroidal axes
(iv) Also deduce all the above results for a uniform circular Lamina of radius a, (V) Find moment of

inertia of circular Lamina about an axis through any point on its circumference and perpendicular to its plane

Sol# Let O be c.m of elliptic Lamin and symmetry axes through be taken as axes.



Dividing disc into strips parallel to YY', consider a such strip PQ at distance x from O and of width. Let ρ be uniform density.

Equation of the boundry of lamina is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y = b/a \sqrt{a^2 - x^2}$$

Area of the thin strip PQ = $2y\,dx$

differential mass of the strip is

$$dm = \rho \cdot 2y\,dx$$

x-axis passes through the centre of this thin rod of length $2y$.

Moment of inertia of rod PQ about x-axis is

$$dI_{xx} = \frac{1}{3} dm \cdot y^2$$

$$= \frac{1}{3} \cdot 2\rho y \cdot y^2 dx \qquad \because dm = 2\rho y\,dx$$

$$= 2/3\, \rho y^3 dx$$

Moment of inertia of disc about x-axis is

$$I_{xx} = \int_{-a}^{a} 2/3\, \rho\, y^3 dx$$

Let $(x, y) = (a\cos\theta, b\sin\theta)$

$\Rightarrow \quad y = b\sin\theta$

& $\quad x = a\cos\theta$

$dx = -a\sin\theta\, d\theta$

When $x = -a \qquad \cos\theta = -1$

$\Rightarrow \theta = \pi$

When $x = a \qquad \cos\theta = 1$

$\Rightarrow \theta = 0$

$$I_{xx} = -\frac{2}{3}\rho \int_0^{\pi} (b\sin\theta)^3 \cdot (a\sin\theta)\, d\theta$$

$$= \frac{2}{3}\rho\, b^3 a \cdot 2 \cdot \int_0^{\pi/2} \sin^4\theta\, d\theta$$

By Wallis formula

$$= \frac{4}{3}\rho\, b^3 a \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a b^3 \rho}{4}$$

Area of elliptic Lamina $= \pi ab$

$$\rho = \frac{\text{Mass}}{\text{Area}} = \frac{M}{\pi ab}$$

$$I_{xx} = \frac{\pi a b^3}{4} \cdot \frac{M}{\pi ab}$$

$$= \frac{Mb^2}{4}$$

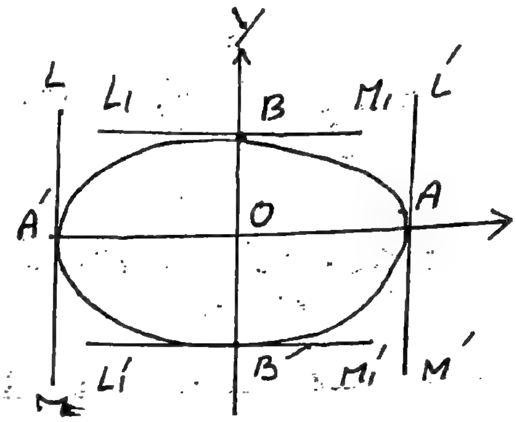
Similarly

$$I_{yy} = \frac{Ma^2}{4}$$

(ii) By theorem of perpendicular axis moment of inertia about an axis

$$\frac{1}{4}Ma^2 + \frac{1}{4}Mb^2 = \frac{1}{4}M(a^2+b^2)$$

(iii) M.I About Tangent at the ends of Centroiadal axes.



Consider tangents LM, $L'M'$ at ends of major axis. Then tangents are $\perp$ar to major axis and hence are parallel to minor axis.

By parallel axes Theorem

$$I_{LM} = I_{L'M'} = I_{yy} + Ma^2$$

$$= \frac{1}{4}Mb^2 + Ma^2 = \frac{5}{4}Ma^2$$

Similarly tangents L_1M_1 & $L_1'M_1'$ at the ends of minor axis are parallel to major axis and

$$I_{L_1M_1} = I_{L_1'M_1'} = I_{xx} + Mb^2$$

$$= \frac{1}{4}Mb^2 + Mb^2 = \frac{5}{4}Mb^2$$

Alternate Method#

Here $dm = \rho \, dx dy$

$$I_{xx} = \iint\limits_{\text{whole Lamina}} y^2 dm = \rho \iint\limits_{\text{Lamina}} y^2 dx dy$$

$$= \rho \int_{-a}^{a} \int_{-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} y^2 dx dy$$

$$= 4\rho \int_0^a dx \left| \frac{y^3}{3} \right|_0^{\frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{4\rho}{3} \int_0^a \frac{b^3}{a^3} (a^2-x^2)^{3/2} dx$$

$$= \frac{4}{3}\rho \cdot \frac{b^3}{a^3} \int_0^a (a^2-x^2)^{3/2} dx$$

let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

When $x = 0$ $\quad \sin\theta = 0 \Rightarrow \theta = 0$

$x = a$ $\quad \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$I_{xx} = \frac{4}{3}\rho \cdot \frac{b^3}{a^3} \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{3/2} \cdot a\cos\theta\, d\theta$$

$$= \frac{4}{3}\rho \cdot \frac{b^3}{a^3} \cdot a^4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{4}{3}\rho\, b^3 a \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} \quad \text{By Walliss formula}$$

$$= \rho \frac{b^3 a \pi}{4} = \frac{M}{\pi ab} \cdot \frac{b^3 a \pi}{4}$$

$$= \frac{1}{4} M b^2$$

Similarly

$$I_{yy} = \frac{1}{4} M a^2$$

(iv) Deductions for Circular Lamina

For elliptic Lamina

$$I_{xx} = \frac{1}{4} M b^2 \quad \rightarrow (1)$$

$$I_{yy} = \frac{1}{4} M a^2 \quad \rightarrow (2)$$

For circular Lamina of radius (a), b = a & we have

$$I_{xx} = \frac{1}{4}Ma^2$$

$$I_{yy} = \frac{1}{4}Ma^2$$

For Polar Moment

$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{1}{4}Ma^2 + \frac{1}{4}Ma^2$$

$$= \frac{1}{2}Ma^2$$

For tangents LM, $L'M'$ at ends of diameter along x-axis

$$I_{LM} = \frac{1}{4}Ma^2 + Ma^2 = \frac{5}{4}Ma^2 = I_{L'M'}$$

About an axis perpendicular to Lamina at any Point of Circumference

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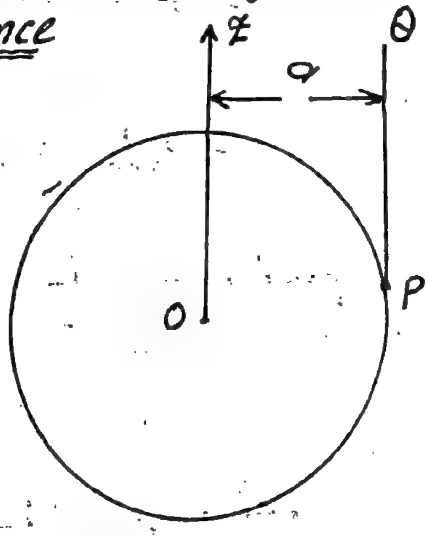


Any axis PQ perpendicular to a point P of circumfirence is parallel to central polar axis OZ and at distance (a) from it. Therefore by parallel axis theorem, we have

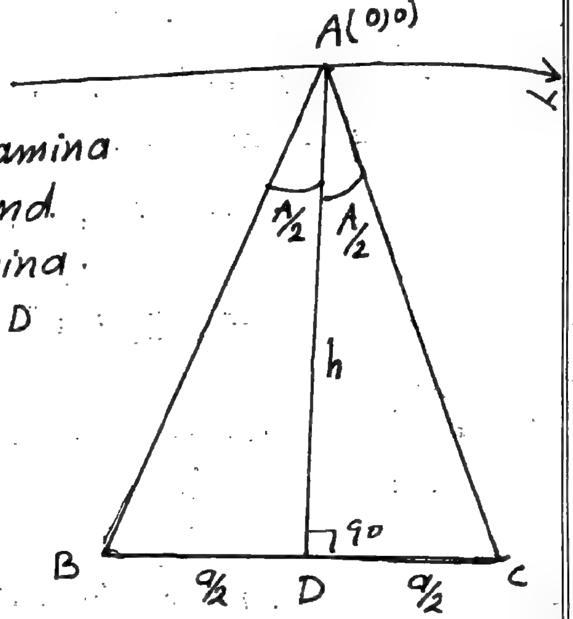
$$I_{PQ} = I_{zz} + Ma^2$$

$$= \frac{1}{2}Ma^2 + Ma^2$$

$$= \frac{3}{2}Ma^2 \quad \text{(Proved)}$$

Triangular Laminas

Question #1: An isosceles triangle Lamina ABC. Find moment of inertia of ABC through the vertex A perpendicular to (i) the opposite side (ii) the plane of the Lamina.



Sol # Since Triangular lamina is isosceles. Let $AB = AC$ and A be the vertex of Lamina. If we drop perpendicular AD on BC, it will necessarily bisect the vertex and side BC. Let ρ be the uniform density of the lamina.
taking AD x-axis and y-axis perpendicular to it.

Area of lamina $= \frac{1}{2}ah$.

Density of Lamina $= \dfrac{M}{\frac{1}{2}ah} = \dfrac{M}{\frac{1}{2}\cdot\frac{a^2}{2}\cot\left(\frac{A}{2}\right)}$

$$\tan\left(\frac{A}{2}\right) = \frac{\frac{a}{2}}{AD} \Rightarrow AD = \frac{a}{2}\cot\left(\frac{A}{2}\right)$$

Equation of line AC is

$$y - 0 = \tan(A/2)(x-0)$$
$$y = x\tan\left(\frac{A}{2}\right)$$

Equation of line AB is

$$y - 0 = \tan(2\pi - A/2)(x-0)$$
$$y = -x\tan\left(\frac{A}{2}\right)$$

Moment of Inertia about AD is

$$I_{AD} = I_{xx} = \iint\limits_{\text{over Lamina}} y^2 \rho \, dx\, dy$$

$$\cancel{= \int_{y=-x\tan(\frac{A}{2})}^{y=x\tan(\frac{A}{2})} \int_0^{\frac{a}{2}\cot(\frac{A}{2})} y^2 \, dx\, dy}$$

$$\cancel{= \int_{-x\tan\frac{A}{2}}^{x\tan\frac{A}{2}} dx \left[\frac{y^3}{3} \right]_{y=0}^{\frac{a}{2}\cot\frac{A}{2}}}$$

$$= \rho \int_0^{\frac{a}{2}\cot\frac{A}{2}} \int_{y=-x\tan\frac{A}{2}}^{y=x\tan(\frac{A}{2})} y^2 \, dy \quad dx$$

$$= 2\rho \int_0^{\frac{a}{2}\cot(\frac{A}{2})} \int_0^{x\tan(\frac{A}{2})} y^2 \, dy \, dx$$

$$= 2\rho \int_0^{\frac{a}{2}\cot(\frac{A}{2})} \left| \frac{y^3}{3} \right|_0^{x\tan(\frac{A}{2})} dx$$

$$= \frac{2\rho}{3} \int_0^{\frac{a}{2}\cot(\frac{A}{2})} x^3 \tan^3\left(\frac{A}{2}\right) dx$$

$$= \frac{2\rho}{3} \tan^3\left(\frac{A}{2}\right) \left| \frac{x^4}{4} \right|_0^{\frac{a}{2}\cot(\frac{A}{2})}$$

$$= \frac{2\rho}{3} \tan^3\left(\frac{A}{2}\right) \cdot \frac{1}{4} \cdot \frac{a^4}{16} \cot^4(A/2)$$

$$= \frac{a^4}{96} \rho \cot\left(\frac{A}{2}\right)$$

putting $\rho = \dfrac{M}{\frac{a^2}{4}\cot\left(\frac{A}{2}\right)}$

$$I_{AD} = I_{xx} = \frac{a^4}{96} \cdot \frac{4 \cdot M}{a^2 \cot(A/2)} \cdot \cot(A/2)$$

$$\boxed{I_{xx} = \frac{Ma^2}{24}}$$

Moment of inertia about y-axis

$$I_{YY} = \iint_{\text{Lamina}} \rho x^2 \, dx \, dy$$

$$= \rho \int_{y=-x\tan\left(\frac{A}{2}\right)}^{y=x\tan(A/2)} \int_0^{\frac{a}{2}\cot\left(\frac{A}{2}\right)} x^2 \, dx \, dy$$

$$= 2\rho \int_0^{a/2\cot(A/2)} x^2 \, dx \int_0^{x\tan(A/2)} dy$$

$$= 2\rho \int_0^{a/2\cot(A/2)} x^2 \left| y \right|_0^{x\tan(A/2)} dx$$

$$= 2\rho \int_{0}^{\frac{a}{2}\cot(A/2)} x^3 \tan(A/2)\, dx$$

$$= 2\rho \tan(A/2) \left| \frac{x^4}{4} \right|_{0}^{\frac{a}{2}\cot(\frac{A}{2})}$$

$$= 2\rho \tan(A/2) \cdot \frac{a^4}{64} \cot^4(\tfrac{A}{2})$$

putting $\rho = \dfrac{M}{\frac{a^2}{4}\cot(A/2)}$

$$I_{yy} = 2 \cdot \frac{M}{\frac{a^2}{4}\cot(A/2)} \cdot \tan(A/2) \cdot \frac{a^4}{64} \cot^4(A/2)$$

$$= \frac{Ma^2}{8} \cot^2(A/2)$$

(ii) Moment of inertia about an axis through A and perpendicular to the plane is

$$I_{AZ} = I_{xx} + I_{yy}$$

$$= \frac{Ma^2}{24} + \frac{Ma^2}{8} \cot^2(A/2)$$

$$= \frac{1}{8} Ma^2 \left(\frac{1}{3} + \cot^2(\tfrac{A}{2}) \right)$$

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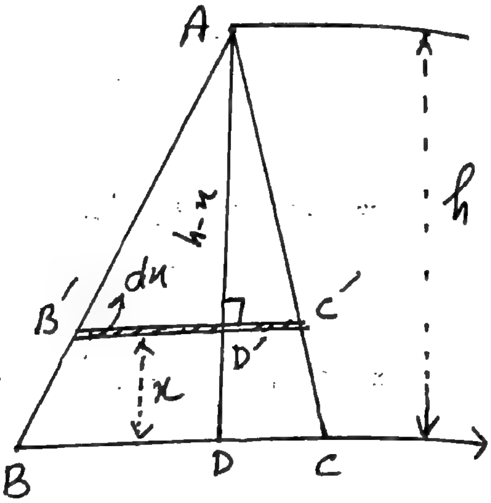
Question # 2 # A uniform triangular lamina ABC. prove that its moment of intertia about BC is $\frac{Mh^2}{6}$, where h is the distance of A from BC



Sol # Let ABC be uniform triangular lamina. h is distance of A from BC. Considering lamina made up of thin strips parallel to BC. Consider a strip $B'C'$ of width dx at distance x from BC.

Now we know that moment of inertia of rod of mass M about an axis at distance d, & parallel to rod is Md^2.

If dm is mass of the strip, then its moment of inertia about BC is

$$dI = x^2 dm.$$

To find dm, we find length of strip:

Consider triangles ABC & AB'C'

$\because$ AC, AC' are colinear, AB', AB are colinear

Also BC, B'C' are parallel.

$\therefore$ triangles are similar and their sides will be proportional

Also ΔACD & $\Delta AC'D'$ are similar

$$\Rightarrow \quad \frac{D'C'}{DC} = \frac{h-x}{h}$$

$$D'C' = \left(\frac{h-x}{h}\right) DC \longrightarrow (1)$$

Similarly ΔABD & $\Delta AB'D'$ are similar

$$\Rightarrow \quad \frac{B'D'}{BD} = \frac{h-x}{h}$$

$$\Rightarrow \quad B'D' = \left(\frac{h-x}{h}\right) BD \longrightarrow (2)$$

Adding ① & ②

$$D'C' + B'D' = \left(\frac{h-x}{h}\right)(DC+BD)$$

$$B'C' = \frac{h-x}{h} \cdot BC$$

$$\therefore B'C' = \left(\frac{h-x}{h}\right)a \quad [\because BC = a]$$

Therefore

dm = mass of the strip

$$= \rho\, B'C' \cdot dx$$

$$= a\rho\left(\frac{h-x}{h}\right)dx$$

where ρ is mass per unit area

$$\therefore dI = a\rho \cdot x^2\left(\frac{h-x}{h}\right)dx$$

$$= \frac{a\rho}{h}(hx^2 - x^3)\,dx$$

Moment of inertia of the lamina about BC

$$= I = \frac{a\rho}{h}\int_0^h (hx^2 - x^3)\,dx$$

$$= \frac{a\rho}{h}\left[\frac{hx^3}{3} - \frac{x^4}{4}\right]_0^h$$

$$= \frac{a\rho}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$= \frac{a\rho}{h}\,\frac{h^4}{12} = \frac{a\rho \cdot h^3}{12}$$

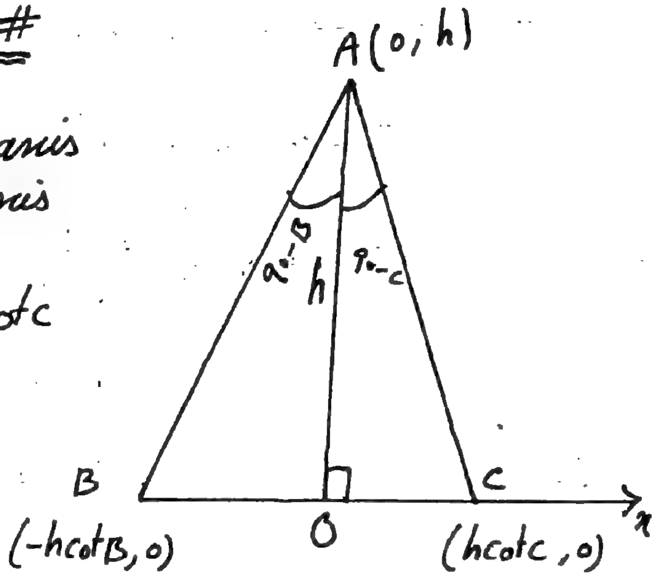
Area of lamina $= \frac{1}{2}ah$

$$\rho = \frac{M}{\frac{1}{2}ah}$$

$$I = \frac{ah^3}{12} \cdot \frac{M}{\frac{1}{2}ah}$$

$$\boxed{I = \frac{1}{6}Mh^2} \Longrightarrow \text{Result}$$

Method-II



Let AO = h.
taking BC along x-axis
and OA = h as y-axis
from Δ AOC

$$\frac{OC}{h} = \tan(90^\circ - C) = \cot C$$

$$OC = h\cot C$$

$\Rightarrow$ Co-ordinates of C are $(h\cot C, 0)$

From Δ AOB

$$\frac{OB}{h} = \tan(90 - B) = \cot B$$

$$\Rightarrow OB = h\cot B$$

$\Rightarrow$ Co-ordinates of B are $(-h\cot B, 0)$

Also

$$BO + OC = h\cot B + h\cot C$$

$$a = h(\cot B + \cot C)$$

$$\frac{a}{h} = \cot B + \cot C \longrightarrow (1)$$

Equation of AB is

$$\frac{y-0}{h} = \frac{x + h\cot B}{h\cot B}$$

$$\Rightarrow \quad y = \frac{x + h\cot B}{\cot B}$$

$$\Rightarrow \quad x = y\cot B - h\cot B \quad \rightarrow (2)$$

Equation of AC

$$\frac{y-0}{h-0} = \frac{x - h\cot C}{-h\cot C}$$

$$y = \frac{x - h\cot C}{-\cot C}$$

$$\Rightarrow \quad x = -y\cot C + h\cot C \quad \rightarrow (3)$$

Moment of inertia about AO is

$$I_{AO} = I_{xx} = \rho \int_{x = y\cot B - h\cot B}^{x = -y\cot C + h\cot C} \int_0^h y^2 \, dy \, dx$$

$$= \rho \int_0^h y^2 \left[x \right]_{y\cot B - h\cot B}^{-y\cot C + h\cot C} dy$$

$$= \rho \int_0^h y^2 \left[-y\cot C + h\cot C - (y\cot B - h\cot B) \right] dy$$

$$= \rho \int_0^h y^2 \left\{ -y(\cot C + \cot B) + h(\cot B + \cot C) \right\} dy$$

$$= \rho \int_0^h y^2 (h - y)(\cot B + \cot C) \, dy$$

By ① $\frac{a}{h} = \cot B + \cot C$

$$I_{OA} = \frac{\rho a}{h}\int_0^h (hy^2 - y^3)\,dy$$

$$= \frac{\rho a}{h}\left| \frac{hy^3}{3} - \frac{y^4}{4} \right|_0^h$$

$$= \frac{\rho a}{h}\left[\frac{h^4}{3} - \frac{h^4}{4} \right] = \frac{\rho a}{h}\left(\frac{h^4}{12}\right)$$

$$= \frac{\rho a h^3}{12}$$

Area of Lamina $= \frac{1}{2}ah$

$$\rho = \frac{M}{\frac{1}{2}ah}$$

$$\boxed{I_{OA} = \frac{M}{\frac{1}{2}ah}\cdot\frac{ah^3}{12} = \frac{1}{6}Mh^2}$$ ⇒ Result

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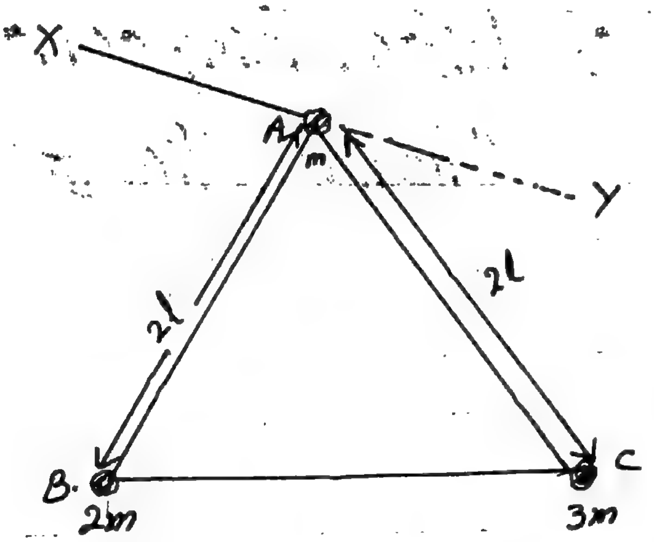
Question # Three light rods, each of length $2l$, are jointed together to form a triangle. Three particles A, B, C of mass $m, 2m, 3m$ are fixed to vertices of the triangle. Find the moment of inertia of the resulting body about

(a) an axis through A perpendicular to the plane ABC,

(b) an axis passing through A and mid point of BC



B is distant $2l$ from the axis XY
so moment of inertia of B about XY is

$$I_B = 2m(2l)^2$$

Similarly

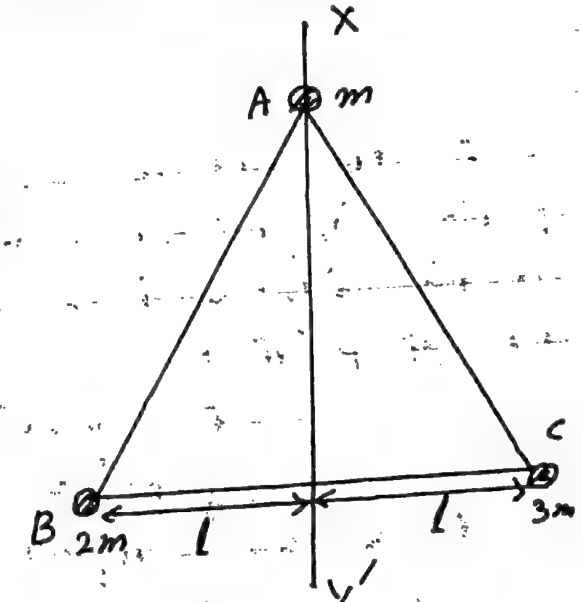
$$I_C = 3m(2l)^2$$

$$I_A = m(0)^2$$

Therefore moment of inertia of the body about XY is

$$I_{XY} = 2m(2l)^2 + 3m(2l)^2 + m(0)^2$$

$$= 20ml^2$$



(b)

I_A about $X'Y' = m(0)^2$

I_B about $X'Y' = 2m(l)^2$

I_C about $X'Y' = 3m(l)^2$

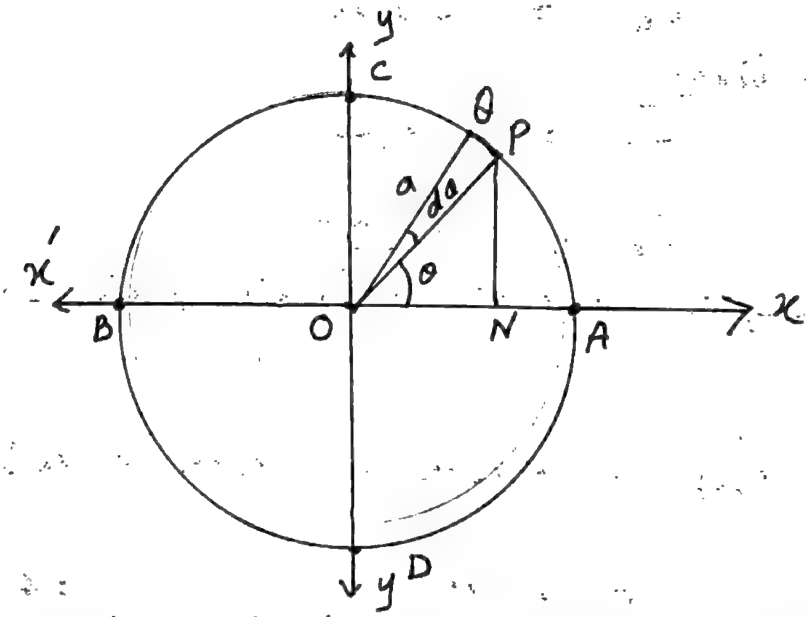
Therefore moment of inertia of the body about $X'Y'$ is

$$I_{X'Y'} = m(0)^2 + 2m(l)^2 + 3ml^2 = 5ml^2$$

M·I of Uniform Ring or HooP or Uniform Circular Wire

Question# Find the moment of Inertia of a uniform ring of mass M and radius 'a' about
(i) a diameter of ring
(ii) a tangent
(iii) an axis through its centre and perpendicular to the plane of ring
(iv) an axis at any point of ring and perpendicular to the plane of ring

Sol #



Let AB and CD be two perpendicular diameters of the ring through centre O. Let P be any point on the ring. Consider a small arc element of mass dm at point P.

$$\text{Let } \angle XOP = \theta$$

$$PN = a \sin\theta$$

Let ρ be uniform density of the ring, then

$$\rho = \frac{M}{\text{Length}} = \frac{M}{2\pi a}$$

Diameter AB is taken along x-axis.

$$ds = \widehat{P\theta} = a\,d\theta$$

$$dm = \rho a\,d\theta$$

Moment of inertia of dm about AB i.e x-axis is

$$dI_{AB} = dI_{xx} = (a^2 \sin^2\theta) \cdot dm$$

$$= a^2 \sin^2\theta \cdot \rho a\,d\theta$$

$$= a^3 \rho \sin^2\theta\, d\theta$$

Moment of Inertia of the whole ring about AB is

$$I_{AB} = I_{xx} = a^3 \rho \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= a^3 \rho \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^3 \rho}{2} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{a^3 \rho}{2} \left[(2\pi - 0) - \frac{1}{2}(\sin 4\pi - \sin 0^\circ)\right]$$

$$= \rho a^3 \pi$$

putting $\rho = \frac{M}{2\pi a}$

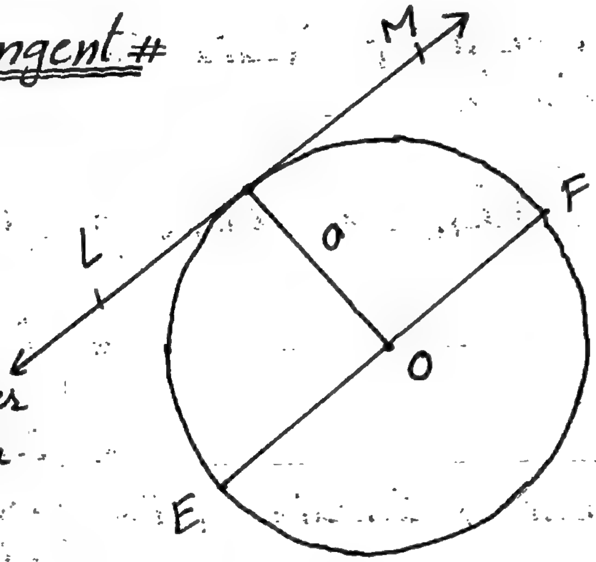
$$I_{AB} = \frac{M}{2\pi a} a^3 \pi = \frac{1}{2} Ma^2$$

$$\boxed{I_{AB} = \frac{1}{2} Ma^2} \longrightarrow \text{Result}$$

By symmetry, we have

$$I_{CD} = I_{yy} = \frac{1}{2} Ma^2$$

(ii) # About tangent #



Let LM be the tangent at any point of the ring and EF be diameter parallel to the tangent. Then

Moment of inertia about diameter is

$$I_{EF} = \frac{1}{2} Ma^2$$

By parallel axes theorem the moment of inertia about tangent LM is

$$I_{LM} = I_{EF} + Ma^2$$

$$= \frac{1}{2} Ma^2 + Ma^2$$

$$= \frac{3}{2} Ma^2$$

$$\boxed{I_{tangent} = \frac{3}{2} Ma^2} \longrightarrow \text{Result.}$$

(iii) About an Axis through centre & Perp to plane of ring

Let OZ be an axis through

centre O of the ring and perpendicular to the plane of ring. Then

$$I_{oz} = I_{xx} + I_{yy}$$

$$= \frac{1}{2}Ma^2 + \frac{1}{2}Ma^2$$

$$= Ma^2$$

OR

directly, we have

$$dI_{oz} = a^2\, dm$$

$$= a^2 \rho a\, d\theta$$

$$= a^3 \rho\, d\theta$$

$$\Rightarrow I_{oz} = a^3\rho \int_0^{2\pi} d\theta = a^3\rho \left|\theta\right|_0^{2\pi}$$

$$= a^3\rho\,(2\pi - 0)$$

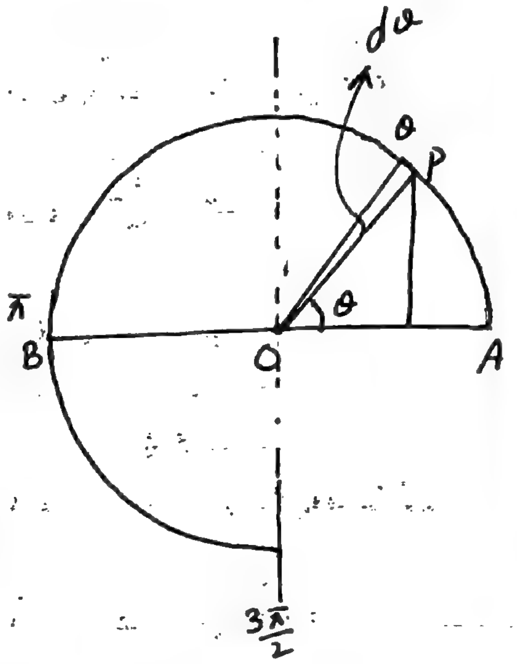
$$= 2a^3\rho \cdot \pi$$

$$= 2a^3 \cdot \frac{M}{2\pi a} \cdot \pi = Ma^2$$

$$\boxed{I_{oz} = Ma^2}$$

Corollary # Find moment of inertia of semi-circular ring about diameter and about an axis through centre and perpendicular to the plane of ring. Also calculate the same for

for one quarter of the ring and for three quarters of the ring.



<u>Sol#</u> $dm = \rho a \, d\theta$

$$dI_{AB} = a^3 \rho \sin^2\theta \, d\theta$$

Moment of inertia of semi-ring about diameter is

$$I_{AB} = a^3 \rho \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= a^3 \rho \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^3 \rho}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi}$$

$$= \frac{a^3}{2} \rho (\pi) = \frac{a^3}{2} \cdot \frac{M}{2\pi a} \cdot \pi$$

$$= \frac{1}{4} M a^2$$ = half of the Moment of inertia of ring about diameter

$$I_{AB} = \frac{1}{4} M a^2$$

Moment of inertia of semi-ring about an axis through centre and perpendicular to the plane of ring is

$$I_{Oz} = \rho a^3 \int_0^{\pi} d\theta = \rho a^3 \pi$$

$$= \frac{1}{2} M a^2$$

= half of the moment of inertia of the ring about the same axis.

For One Quarter of Ring

The moment of inertia of one quarter of ring about diameter is

$$I_{AB} = a^3 \rho \int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= a^3 \frac{M}{2\pi a} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \qquad \text{Wallis formula}$$

$$= \frac{1}{8} M a^2$$

= One quarter of the moment of inertia of the ring about diameter

Wallis formula $\int_0^{\pi/2} (\sin^n\theta \text{ or } \cos^n\theta) \, d\theta$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3} \qquad \text{when } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \qquad n \text{ is even}$$

The moment of inertia of on quarter of ring about an axis through centre and perpendicular to the plane of ring is

$$I_{oz} = \rho a^3 \int_0^{\pi/2} d\theta = \rho a^3 \left| \theta \right|_0^{\pi/2}$$

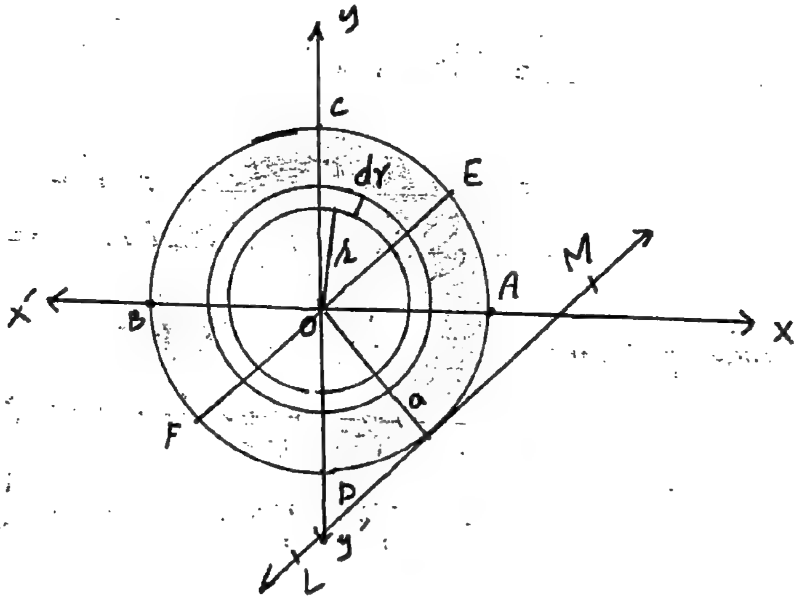
$$= \rho a^3 \cdot \frac{\pi}{2} = \frac{M}{2\pi a} \cdot a^3 \cdot \frac{\pi}{2}$$

$$I_{oz} = \frac{1}{4} M a^2$$

M.I of Uniform Circular & Semi-Circular Disc

Question # Find the moment of inertia of a uniform circular disc of radius a about
(i) # a diameter of the disc
(ii) # a tangent
(iii) an axis through the centre of disc and perpendicular to the plane of disc
(iv) # an axis through any point of circumference and perpendicular to the plane of disc

Sol #



Let O be the centre of disc and AB along x-axis & CD along y-axis be two perpendicular diameters. Let Oz be 3rd orthogonal axis.

Divide the disc into elements in the form of concentric rings. The thickness of disc is taken negligible as compared to its radius.

Consider a typical thin ring of radius r. If ρ is density of the disc, then the mass of typical ring of radius r is

$$dm \approx \rho\,(\text{Area of ring})$$
$$= \rho\,(\text{length} \times \text{breadth})$$
$$= \rho\,(2\pi r \times dr)$$
$$= 2\rho\pi r\,dr$$

The M.I of the ring about AB (diameter) is by previous article

$$dI_{AB} = dI_{xx} = \frac{1}{2} r^2\,dm$$
$$= \frac{1}{2} r^2 \cdot 2\rho\pi r\,dr$$
$$= \rho\pi r^3\,dr$$

$\Rightarrow$ M.I of the disc about AB (x-axis) is

$$I_{AB} = I_{xx} = \rho\pi \int_0^a r^3\,dr$$
$$= \rho\pi \left| \frac{r^4}{4} \right|_0^a$$
$$= \frac{\rho\pi}{4} \cdot a^4$$

$$\rho = \frac{M}{\text{area of disc}} = \frac{M}{\pi a^2}$$

$$I_{AB} = \frac{M}{\pi a^2} \cdot \frac{\pi}{4} \cdot a^4$$

$$\boxed{I_{AB} = M \cdot \frac{a^2}{4}} \longrightarrow \text{Result}$$

Similarly by symmetry, we have

$$I_{CD} = \frac{1}{4} M a^2$$

(ii) By Theorem of parallel axes, the moment of

of inertia about a tangent LM is calculated as

Let EF be diameter parallel to LM.

Then

$$I_{EF} = \frac{1}{4} Ma^2$$

$$I_{LM} = I_{EF} + Ma^2$$

$$= \frac{1}{4} Ma^2 + Ma^2$$

$$\boxed{I_{LM} = \frac{5}{4} Ma^2} \longrightarrow \text{Result}$$

(iii) About Centroidal axis Perpendicular to disc#

By perpendicular axes theorem, we have

$$I_{OZ} = I_{xx} + I_{yy}$$

$$= \frac{1}{4} Ma^2 + \frac{1}{4} Ma^2$$

$$\boxed{I_{OZ} = \frac{1}{2} Ma^2} \longrightarrow \text{Result}$$

(IV) About an axis through Circumference and Perp. to dis#

Such an axis will be parallel to centroidal perpendicular axis OZ and at distance (a) from it. Therefor by parall axes Theorem moment of inertia about PZ′ (PZ′ ∥ OZ) is

$$I_{PZ'} = I_{OZ} + Ma^2$$

$$= \frac{1}{2} Ma^2 + Ma^2 = \frac{3}{2} Ma^2$$

Direct Method#

(i) **About Diameters#**

$$I_{AB} = I_{xx} = \int_{Disc} y^2 \, dm$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2 \rho \, dx\, dy$$

$$= \rho . 4 \int_0^{\sqrt{a^2-x^2}} \int_0^{a} y^2 \, dx\, dy$$

$$= 4\rho \int_0^{a} \left| \frac{y^3}{3} \right|_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{4}{3}\rho \int_0^{a} (a^2-x^2)^{3/2} dx$$

Let $x = a\sin\theta$

$dx = a\cos\theta \, d\theta$

When $x = 0$ $\quad \sin\theta = 0 \Rightarrow \theta = 0$

$x = a$ $\quad \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$I_{AB} = I_{xx} = \frac{4\rho}{3} \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{3/2} . \, a\cos\theta \, d\theta$$

$$= \frac{4\rho}{3} . a^4 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{4\rho}{3} . a^4 . \frac{3.1}{4.2} . \frac{\pi}{2} = \rho a^4 . \frac{\pi}{4}$$

Putting $\rho = \frac{M}{\pi a^2}$

$$I_{AB} = \frac{M}{\pi a^2} \cdot \frac{\pi}{4} a^4$$

$$\boxed{I_{AB} = \frac{1}{4} M a^2} \longrightarrow \text{Result.}$$

Similarly $I_{yy} = \frac{1}{4} M a^2$

<u>About Centroidal axis perpendicular to Disc #</u>

$$I_{zz} = \rho \iint_{\text{Disc}} (x^2+y^2)\, dx\, dy$$

$$= \rho \int_0^{2\pi}\int_0^{a} r^2 \cdot r\, dr\, d\theta$$

$$= \rho \int_0^{2\pi}\int_0^{a} r^3\, dr\, d\theta$$

$$= \rho \int_0^{2\pi} \left|\frac{r^4}{4}\right|_0^a d\theta$$

$$= \frac{\rho a^4}{4}\int_0^{2\pi} d\theta = \frac{a^4 \rho}{4} 2\pi$$

$$= \frac{a^4 \rho \pi}{2} = \frac{a^4 \pi}{2} \cdot \frac{M}{\pi a^2}$$

$$\boxed{I_{zz} = \frac{1}{2} M a^2} \longrightarrow \text{Result.}$$

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Uniform Hollow Right Circular Cylinder

Question# Find the moment of inertia of a uniform right circular hollow cylinder of mass M, radius r and height (length) h about

(i) # the axis of cylinder

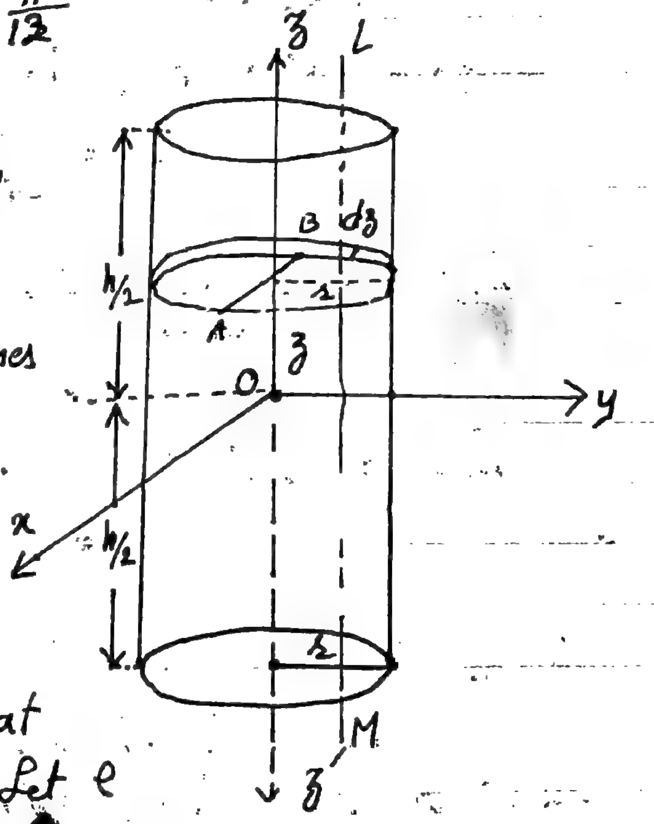
(ii) # an axis through c.m of cylinder and perp. to axis of the cylinder

(iii) # an element of the cylinder

(IV) # a diameter of one end

(V) # a tangent line which lies in a plane passing through the centre of mass and perpendicular to axis and prove that radius of gyration k about this tangent is given by

$$k^2 = \frac{3r^2}{2} + \frac{h^2}{12}$$



Sol# Divide the cylinder into uniform circular ring all parallel to the ends of cylinder or with planes perpendicular to the axis zz' of the cylinder. O is centre of mass of the cylinder.

Consider a circular ring of breadth dz at distance z from O. Let ρ be areal density of cylinedr.

Area of circular ring $= 2\pi r\, dz$

Mass of the ring is

$$dm = \rho \cdot 2\pi r\, dz$$

For the ring element axis zz' is centroidal axis perpendicular to its plane.

The moment of inertia of ring about zz' is

$$dI_{zz'} = r^2 dm$$

$$= r^2 \cdot \rho \cdot 2\pi r dz$$

$$= 2\rho\pi r^3 dz$$

$$\Rightarrow \quad I_{zz'} = 2\rho\pi r^3 \int_{-h/2}^{h/2} dz$$

$$= 2\rho\pi r^3 \cdot [z]_{-h/2}^{h/2} = 2\rho\pi r^3 \left[\frac{h}{2} + \frac{h}{2}\right]$$

$$= 2\rho\pi r^3 \cdot h$$

Surface area of cylinder $= 2\pi r \cdot h$.

$$\rho = \frac{M}{2\pi r h}$$

$$I_{zz'} = 2 \cdot \frac{M}{2\pi r h} \cdot \pi r^3 h = M r^2$$

$$\boxed{I_{zz'} = M r^2} \longrightarrow \text{Result}$$

(ii) <u>About an axis through c.m & Perp. to axis #</u>

The axis through c.m and perpendicular to the axis of cylinder will be parallel to some diameter of the ring element and diameter of the ring element is centroidal axis for ring. Let Ox be axis through c.m and $\perp$ar to the axis of cylinder and AB be

diameter of ring parallel to OX.

Moment of inertia of ring about diameter AB is

$$dI_{AB} = \frac{1}{2} r^2 dm$$

$$= \frac{1}{2} r^2 \cdot \rho \cdot 2\pi r dz$$

$$= \rho \cdot \pi r^3 dz$$

By parallel axes theorem moment of inertia of ring abot OX is

$$dI_{Ox} = dI_{AB} + z^2 dm$$

$$= \rho \pi r^3 dz + z^2 \cdot \rho \cdot 2\pi r dz$$

$$I_{Ox} = \rho \pi r^3 \int_{-h/2}^{h/2} dz + \rho \cdot 2\pi r \int_{-h/2}^{h/2} z^2 dz$$

$$= \rho \pi r^3 \left| z \right|_{-h/2}^{h/2} + \rho \cdot 2\pi r \left| \frac{z^3}{3} \right|_{-h/2}^{h/2}$$

$$= \rho \pi r^3 \left[\frac{h}{2} + \frac{h}{2} \right] + \frac{\rho \cdot 2\pi r}{3} \left[\frac{h^3}{8} + \frac{h^3}{8} \right]$$

$$= \rho \pi r^3 h + \frac{2}{3} \pi r \cdot \rho \cdot \frac{h^3}{4}$$

$$= \rho \pi r h \left(r^2 + \frac{2h^2}{12} \right)$$

putting $\rho = \dfrac{M}{2\pi r \cdot h}$

$$I_{Ox} = \frac{M}{2\pi r h} \cdot \pi r h \left(r^2 + \frac{2h^2}{12} \right)$$

$$I_{Ox} = M \left(\frac{r^2}{2} + \frac{h^2}{12} \right)$$

$$\boxed{I_{ox} = M\left(\frac{r^2}{2} + \frac{h^2}{12}\right)} \longrightarrow \text{Result}$$

(iii) **About an element of Cylinder #**

Any line lying on the cylinder and parallel to the axis of the cylinder is called an element of the cylinder.

Let LM be an element of the cylinder. Then LM is parallel to the axis of the cylinder and at distance r from axis. Therefor by parallel axis theorem, we have

$$I_{LM} = I_{zz'} + Mr^2$$

$$= Mr^2 + Mr^2 = 2Mr^2$$

$$\boxed{I_{LM} = 2Mr^2} \longrightarrow \text{Result}$$

(iv) **About a diameter of one End#**

Diameter of one end will be parallel to some centroidal axis perpendicular to the axis of cylinder and at distance $h/2$ from it.

Let (d) be a diameter parallel to the centroidal axis, ox. Then by parallel axes theorem, we have

$$I_d = I_{ox} + M\left(\frac{h}{2}\right)^2$$

$$= M\left(\frac{r^2}{2} + \frac{h^2}{12}\right) + M \cdot \frac{h^2}{4}$$

$$I_d = M\left(\frac{r^2}{2} + \frac{h^2}{12} + \frac{h^2}{4}\right)$$

$$I_d = M\left(\frac{r^2}{2} + \frac{h^2}{3}\right)$$

Moment of inertia about diameter of one end is

$$\boxed{I_d = M\left(\frac{r^2}{2}+\frac{h^2}{3}\right)} \longrightarrow \text{Result.}$$

(V) <u>About tangent in Centroidal plane ⊥ar to axis #</u>

tangent to the cylinder, which lies in a centroidal plane ⊥ar to axis (or parallel to one end) will be be parallel to a centroidal axis perpendicular to the axis of the cylinder.

let ox be parallel to tangent line. Then by parallel axis Theorem.

$$I_{tangent} = I_{ox} + Mr^2$$

$$= M\left(\frac{r^2}{2}+\frac{h^2}{12}\right) + Mr^2$$

$$\boxed{I_{tangent} = M\left(\frac{3}{2}r^2+\frac{h^2}{12}\right)} \longrightarrow \text{Result}$$

Thus radius of Gyration about tangent is given by

$$k^2 = \frac{3}{2}r^2 + \frac{h^2}{12}$$

<u>M.I of Uniform Solid Right Circular Cylinder #</u>

Question# Find the moment of inertia of uniform solid right cylinder of radius a, height h, and mass M about

(i) # axis of the cylinder

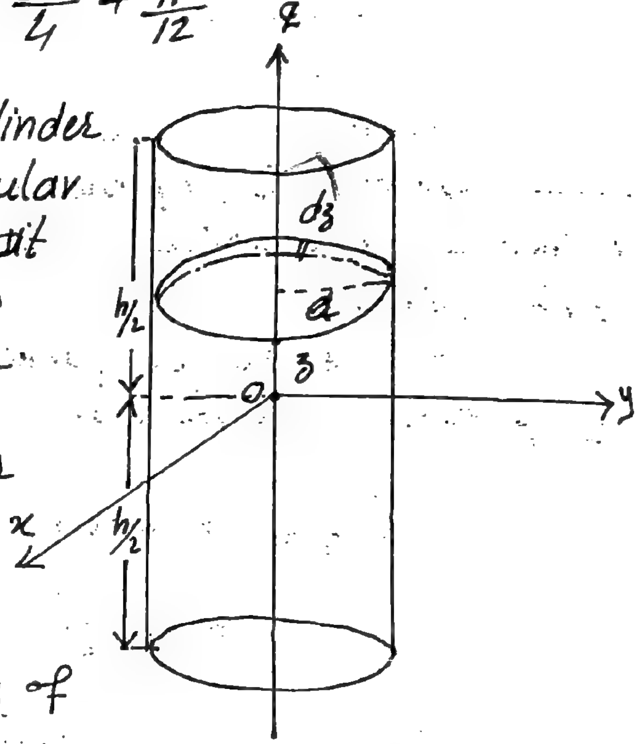
(ii) # Centroidal axis perpendicular to the axis of cylinder

(iii) a generator (element) of cylinder

(iv) # a diameter of one end

(v) # a tangent line which lies in a plane passing through c.m and perpendicular to axis and prove that the radius of gyration K about this tangent is given by

$$K^2 = \frac{5a^2}{4} + \frac{h^2}{12}$$



Sol# Divide the cylinder into uniform circular discs all concentred wit axis and parallel to the ends of cylinder. Let O be c.m of the cylinder. Consider a circular disc of thickness dz at distance z from O. Let ρ be the density of cylinder.

Volume of disc element $= \pi a^2 dz$

Mass of disc element $= \rho\pi a^2 dz = dm$

z-axis is centroidal axis of disc element and perpendicular to the plane of disc.

So Moment of inertia of disc element about axis of the cylinder is

$$dI_{zz} = \frac{1}{2} dm\, a^2 \qquad \text{(From article of disc)}$$

$$= \frac{1}{2}\rho\pi a^4 dz$$

Moment of inertia of the cylinder about axis is

$$I_{zz} = \frac{1}{2}\rho\pi a^4 \int_{-h/2}^{h/2} dz$$

$$I_{zz} = \frac{1}{2}\rho\pi a^4 |z|_{-h/2}^{h/2}$$

$$= \frac{1}{2}\rho\pi a^4 \left[\frac{h}{2}+\frac{h}{2}\right]$$

$$= \frac{1}{2}\rho\pi a^4 h$$

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\pi a^2 h}.$$

$$I_{zz} = \frac{1}{2} \cdot \frac{M}{\pi a^2 h} \cdot \pi a^4 h.$$

$$\boxed{I_{zz} = \frac{1}{2} M r^2} \longrightarrow \text{Result}$$

<u>OR</u>

We can find moment of inertia about axis directly as

$$I_{zz} = \int \rho (x^2+y^2)\, dV$$

We use cylinderical polar Co-ordinates.

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dv = r\,dr\,d\theta\,dz$$
$$0 \le r \le a \qquad 0 \le \theta \le 2\pi \qquad -h/2 \le z \le h/2$$

$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$I_{zz} = \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^a r^2 \cdot r \cdot dr\, d\theta\, dz$$

$$= \rho \int_0^a r^3 dr \int_0^{2\pi} d\theta \int_{-h/2}^{h/2} dz$$

$$= e\frac{a^4}{4}\cdot 2\pi\cdot\left[\frac{h}{2}+\frac{h}{2}\right]$$

$$= \frac{1}{4}e\cdot a^4\cdot 2\pi\cdot h = \frac{1}{2}e\cdot a^4\pi h.$$

putting $e = \frac{M}{\pi a^2 h}$.

$$I_{zz} = \frac{1}{2}Ma^2.$$

(ii) **About Centroidal Axis Perp. to Axis #**

Here OX & OY can be taken as centroidal axis (or ~~dimeteric axis~~) per- to axis

∴ Moment of inertia of disc element about OX (~~diameter~~) can be calculated as.

Let AB be diameter (centroidal axis) of disc element. Then moment of inertia of disc element is

$$dI_{AB} = \frac{1}{4}dm\,a^2$$

$$= \frac{1}{4}a^2\cdot e\,\pi a^2 dz$$

$$= \frac{1}{4}a^4 e\pi\, dz$$

By parallel axes theorem moment of inertia disc element about OX is

$$dI_{OX} = dI_{AB} + z^2 dm$$

$$= \frac{1}{4}a^4 e\pi\, dz + e\pi a^2\cdot z^2 dz$$

Moment of inertia of cylinder about OX is

$$I_{OX} = \frac{1}{4}a^4 e\pi\int_{-h/2}^{h/2} dz + e\pi a^2\int_{-h/2}^{h/2} z^2 dz$$

$$= \frac{1}{4}a^4 \rho \pi |z|_{-h/2}^{h/2} + \rho \pi a^2 \left| \frac{z^3}{3} \right|_{-h/2}^{h/2}$$

$$= \frac{1}{4}a^4 \rho \pi \left[\frac{h}{2} + \frac{h}{2}\right] + \rho \pi a^2 \left[\frac{h^3}{24} + \frac{h^3}{24}\right]$$

$$= \frac{1}{4}a^4 \rho \pi \cdot h + \rho \pi a^2 \cdot \frac{h^3}{12}.$$

$$= \frac{1}{4} \rho \pi a^2 h \left[a^2 + \frac{h^2}{3}\right]$$

putting $\rho = \dfrac{M}{\pi a^2 h}$

$$I_{ox} = \frac{1}{4} \cdot \frac{M}{\pi a^2 h} \cdot \pi a^2 h \left[a^2 + \frac{h^2}{3}\right]$$

$$\boxed{I_{ox} = M\left[\frac{a^2}{4} + \frac{h^2}{12}\right]} \longrightarrow \text{Result.}$$

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Similarly

$$\boxed{I_{oy} = M\left[\frac{a^2}{4} + \frac{h^2}{12}\right]} \longrightarrow \text{Result.}$$

OR

We can calculate moment of inertia about ox & oy directly as

$$I_{ox} = \int \rho (y^2 + z^2)\, dV$$

using cylinderical polar Co-ordinates.

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$$y^2 + z^2 = r^2\sin^2\theta + z^2$$

$$dV = r\,dr\,d\theta\,dz$$

$$I_{ox} = \rho \int_{-h/2}^{h/2} \int_{0}^{2\pi} \int_{0}^{a} (r^2 \sin^2\theta + z^2)\, r\, dr\, d\theta\, dz$$

$$= \rho \int_{-h/2}^{h/2} \int_{0}^{2\pi} \int_{0}^{a} r^3 \sin^2\theta\, dr\, d\theta\, dz$$

$$+ \rho \int_{-h/2}^{h/2} \int_{0}^{2\pi} \int_{0}^{a} r z^2\, dr\, d\theta\, dz$$

$$= \rho \int_{-h/2}^{h/2} dz \int_{0}^{2\pi} \sin^2\theta\, d\theta \int_{0}^{a} r^3\, dr$$

$$+ \rho \int_{0}^{a} r\, dr \cdot \int_{0}^{2\pi} d\theta \int_{-h/2}^{h/2} z^2\, dz$$

$$= \rho \left[z \right]_{-h/2}^{h/2} \cdot \left[\frac{r^4}{4} \right]_{0}^{a} \int_{0}^{2\pi} \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$+ \rho \left[\frac{r^2}{2} \right]_{0}^{a} \left[\theta \right]_{0}^{2\pi} \left[\frac{z^3}{3} \right]_{-h/2}^{h/2}$$

$$= \rho \left[h/2 + h/2 \right] \frac{a^4}{4} \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_{0}^{2\pi}$$

$$+ \rho \left[\frac{a^2}{2} \right] \left[2\pi - 0 \right] \cdot \frac{1}{3} \left[\frac{h^3}{8} + \frac{h^3}{8} \right]$$

$$= \rho \cdot h \cdot \frac{a^4}{8} \left[2\pi \right] \cdot \rho \cdot \frac{a^2}{2} \cdot 2\pi \cdot \frac{1}{3} \cdot \frac{h^3}{4}$$

$$= \rho \frac{h a^4 \pi}{4} + \frac{\rho a^2 h^3 \pi}{12} = \frac{1}{4} \rho \pi a^2 h \left[a^2 + \frac{h^2}{3} \right]$$

putting $\rho = \frac{M}{\pi a^2 h}$

$$I_{ox} = \frac{1}{4} \frac{M}{\pi a^2 h} \cdot \pi a^2 h \left[a^2 + \frac{h}{3}\right]$$

$$I_{ox} = \frac{1}{4} M \left[a^2 + \frac{h}{3}\right]$$

Similarly

$$I_{oy} = \frac{1}{4} M \left[a^2 + \frac{h}{3}\right]$$

(iv) # About Diameter of on End

Diameter of any flat end of the cylinder will be parallel to some centroidal axis $\perp$ar to axis of the cylinder and at distance $h/2$ from it.

Let (d) be diameter of base end parallel to centroidal ox. Then by parallel axes Theorem

$$I_d = I_{ox} + M\left(\frac{h}{2}\right)^2$$

$$= M\left(\frac{a^2}{4} + \frac{h^2}{12}\right) + M\frac{h^2}{4}$$

$$= M\left(\frac{a^2}{4} + \frac{h^2}{12} + \frac{h^2}{4}\right)$$

$$= M\left(\frac{a^2}{4} + \frac{h^2}{3}\right)$$

$$\boxed{I_d = M\left(\frac{a^2}{4} + \frac{h^2}{3}\right)} \longrightarrow \text{Result.}$$

By M. Hussain Lecturer (Math)

(vi) # About a Generator of Cylinder

Any line on cylinder and parallel to the axis of the cylinder is called generator of the cylinder

Let LM be a generator of the cylinder. Then LM is parallel to the axis of the cylinder and at distance (a) from the axis. Therefore by parallel axes theorem, we have

$$I_{LM} = I_{zz} + Ma^2$$

$$= \frac{1}{2}Ma^2 + Ma^2$$

$$\boxed{I_{LM} = \tfrac{3}{2} Ma^2} \longrightarrow \text{Result.}$$

(V) About a tangent in centroidal Plane parallel to ends#

Tangent to the cylinder, which lies in centroidal plane $\perp$ar to the axis of cylinder (or parallel to the ends of cylinder) will be parallel to a centroidal axis $\perp$ar to the axis of the cylinder.

Let ox be parallel to tangent line. then by parallel axes theorem, we have

$$I_{tangent} = I_{ox} + Ma^2$$

$$= M\left(\frac{a^2}{4} + \frac{h^2}{12}\right) + Ma^2$$

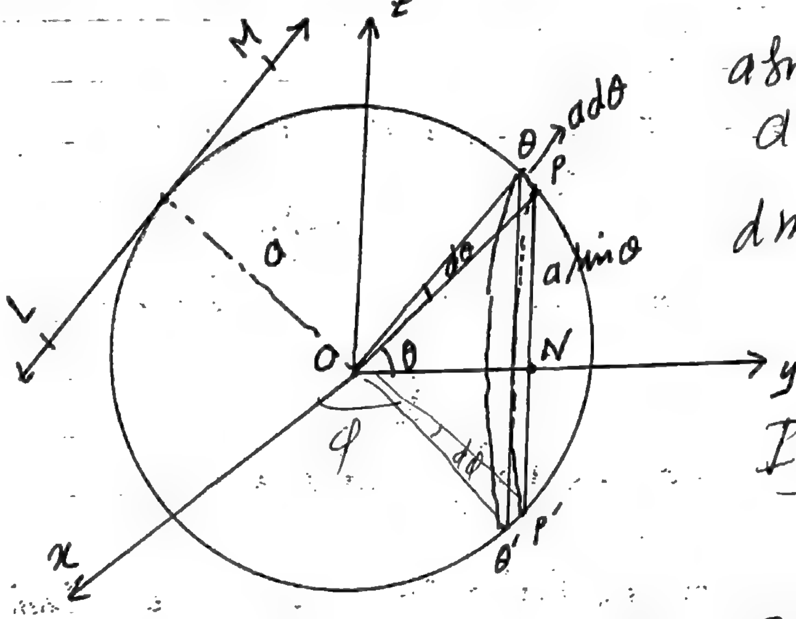
$$\boxed{= M\left(\frac{5a^2}{4} + \frac{h^2}{12}\right)} \longrightarrow \text{Result}$$

Thus radius of Gyration about this tangent is given by $k^2 = \frac{5a^2}{4} + \frac{h^2}{12}$ Proved

M·I of Uniform Hollow Sphere OR Uniform Spherical Shell

Question # Find the moment of inertia of a uniform hollow sphere (spherical shell) of radius (a) and Mass of M about
(i)# a diameter (or symmetry axis)
(ii)# a tangent at any point of shell.

Sol #



$a \sin\theta \, d\theta$
$a \, d\phi$
$dm = \rho a^2 \sin\theta \, d\phi \times d\theta$

$I_y = \int_{Sphere} dm (a \sin\theta)^2$

Let O be the centre of spherical shell. Take axes OX, OY, OZ at O. Divide the sphere into elements by parallel cuts perpendicular to OY. Each element is approximately a circular band (or circular cylinder) and OY is its centroidal axis perpendicular to its diameter. Consider $P\theta P'\theta'$ such circular band.

Radius of circular slice $= a \sin\theta$
Breadth of " " $= a \, d\theta$
Area of circular band $= 2\pi a \sin\theta \cdot a \, d\theta$
$= 2\pi a^2 \sin\theta \, d\theta$
Mass of slice $= dm = \rho \cdot 2\pi a^2 \sin\theta \, d\theta$
The moment of inertia of slice about OY.

$$dI_{yy} = \text{Mass} \times (\text{radius})^2$$

(By method of uniform circular ring about centroidal axis ⊥ar to the plane of ring)

$$dI_{yy} = 2\rho\pi a^2 \sin\theta \, d\theta \cdot a^2 \sin^2\theta$$

$$= 2\rho\pi a^4 \sin^3\theta \, d\theta$$

Hence M.I of hollow sphere about OX, diameter is

$$I_{yy} = \int_0^{\pi} 2\rho\pi a^4 \sin^3\theta \, d\theta$$

$$= 2\rho\pi a^4 \int_0^{2(\pi/2)} \sin^3\theta \, d\theta$$

If $f(2a-x) = f(x)$ then $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$

$$= 4\rho\pi a^4 \int_0^{\pi/2} \sin^3\theta \, d\theta$$

$$= 4\rho\pi a^4 \cdot \frac{2}{3}$$ By Wallis formula

Surface area of sphere $= 4\pi a^2$

$$\rho = \frac{M}{4\pi a^2}$$

$$I_{yy} = 4 \cdot \frac{M}{4\pi a^2} \cdot a^4 \cdot \frac{2}{3}$$

$$I_{yy} = \frac{2}{3} M a^2$$

Thus moment of inertia of hollow sphere about diameter is

$$\boxed{I_{diamete} = \tfrac{2}{3} Ma^2} \longrightarrow \text{Result.}$$

(ii) # <u>About Tangent</u> #

tangent at any point of sphere will be parallel to some diameter of the sphere and diameter of sphere is centroidal axis. So we may apply parallel axes theorem.

Let LM be any tangent and (d) be diameter parallel to LM. By parallel axes theorem, we have

$$I_{LM} = I_d + Ma^2$$

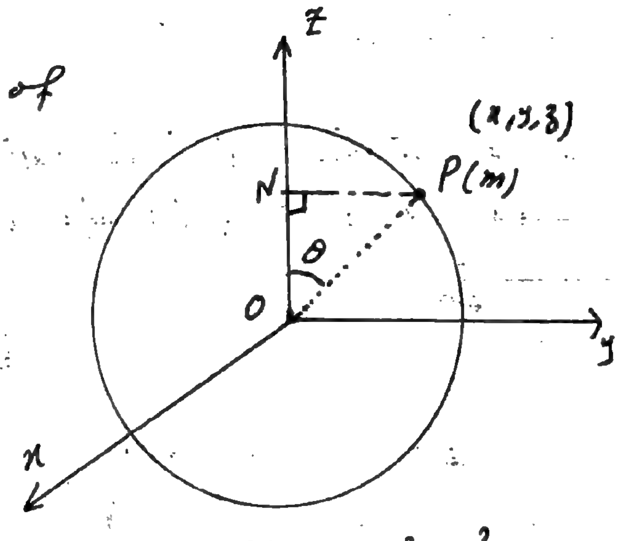
$$= \tfrac{2}{3} Ma^2 + Ma^2$$

$$= \tfrac{5}{3} Ma^2$$

So

$$\boxed{I_{tangent} = \frac{5}{3} Ma^2} \longrightarrow \text{Result.}$$

<u>Method-II</u>



Let P be a particle of shell of mass m and P have co-ordinates (x,y,z).

$$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$PN = |\vec{OP}| \sin\theta$$

$$= |\hat{k} \times \vec{OP}|$$

$$= |\hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k})| = |x\hat{j} - y\hat{i}| = x^2 + y^2$$

Moment of inertia of m about OZ is

$$mNP^2 = m(x^2 + y^2)$$

So the M.I I of shell about OZ is

$$I = \Sigma m(x^2+y^2) \longrightarrow ①$$

By symmetry moment of inertia about OX, OY is also equal to I. So

$$I = \Sigma m(y^2+z^2) \longrightarrow ②$$

$$I = \Sigma m(x^2+z^2) \longrightarrow ③$$

Adding ① & ② & ③

$$3I = \Sigma m(2x^2+2y^2+2z^2)$$

$$= 2(\Sigma m)\cdot(x^2+y^2+z^2)$$

$$= 2Ma^2$$

$$\boxed{I = \frac{2}{3}Ma^2} \Longrightarrow \text{Result}$$

For Hemispherical Shell#

For hemi-spherical shell limits of integration are from 0 to $\pi/2$ and Moment of inertia about symmetry axis oy is I_{oy}

$$I_{oy} = 2\rho\pi a^4 \int_0^{\pi/2} \sin^3\theta \, d\theta$$

$$= 2 \cdot \frac{M}{2\pi a^2} \cdot \pi a^4 \cdot \frac{2}{3}$$ By Wallis formula

$$= \frac{2}{3}Ma^2 \qquad \rho = \frac{M}{2\pi a^2}$$

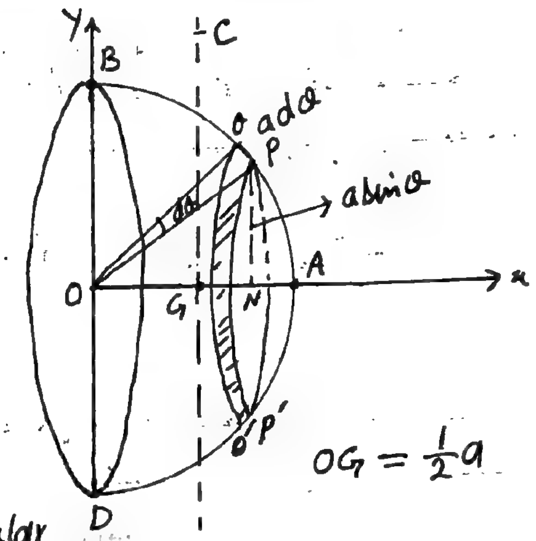
M·I of Hemi-Spherical Spherical Shell

Question# Find the moment of inertia of uniform hemi-spherical shell of mass M, radius a, about

(i) axis of symmetry (ii) a diameter of plane base

(iii)# an axis through its mass centre and perpendicular to the axis of symmetry.

(iiii) a tangent at the end of symmetry axis



Sol# Let O be the centre of the plane base and OA the axis of symmetry.
Let G be its mass-centre. Then

$$OG = \frac{1}{2}a$$

Let BD be diameter of plane base perpendicular to the axis of symmetry and GC be the axis through G parallel to BD.

Divide the sphere into circular slices perp to symmetry axis and centred at the axis.

Consider such a slice of width $ad\theta$.

Radius of the slice $= PN = a\sin\theta$

Area of the slice $= 2\pi \cdot a\sin\theta \cdot ad\theta$

$$= 2\pi a^2 \sin\theta\, d\theta$$

Mass of the slice $= \rho \cdot 2\pi a^2 \sin\theta\, d\theta = dm$.

The symmetry axis OA of the shell is the centroidal axis of the slice and $\perp$ar to its plane.

Moment of inertia of slice about symmetry axis is

$$dI_{ox} = \text{Mass} \times (\text{radius})^2$$

(By method of uniform ring about centroidal axis perpendicular to the plane of ring)

$$dI_{ox} = 2\rho\pi a^2 \sin\theta \cdot d\theta \cdot a^2 \sin^2\theta$$

$$= 2\rho\pi a^4 \sin^3\theta \cdot d\theta$$

Moment of inertia of shell about symmetry axis is

$$I_{ox} = 2\rho\pi a^4 \int_0^{\pi/2} \sin^3\theta \, d\theta$$

$$= 2 \cdot \frac{M}{2\pi a^2} \cdot \pi a^4 \cdot \frac{2}{3} \quad \text{By Wallis formula}$$

$$I_{ox} = \frac{2}{3} M a^2$$ which is same as for spherical shell of radius a.

(ii) <u>About Diameter of Plane Base #</u>

The moment of inertia of circular slice about its own diameter d, parallel to diameter BD of plane base is

$$dI_d = \frac{1}{2} (\text{Mass}) \cdot (\text{radius})^2$$
$$= \frac{1}{2} dm \cdot a^2 \sin^2\theta$$

$$dI_d = \frac{1}{2}\cdot(\rho\cdot 2\pi a^2 \sin\theta\cdot d\theta)\cdot(a^2\sin^2\theta)$$

$$= \rho\pi a^4 \sin^3\theta\, d\theta$$

Now diameter of slice is its centroidal axis and we may apply parallel axes theorem to find its M.I about BD.

$$dI_{BD} = dI_d + (dm)\cdot(a^2\cos^2\theta)$$

$$= \rho\pi a^4 \sin^3\theta\, d\theta + \rho\cdot 2\pi a^2 \sin\theta\cdot d\theta \times a^2\cos^2\theta$$

$$= \rho\pi a^4 \sin^3\theta\, d\theta + 2\rho\pi a^4 \sin\theta \cos^2\theta\, d\theta$$

$$I_{BD} = \rho\pi a^4 \int_0^{\pi/2} \sin^3\theta\, d\theta + 2\rho\pi a^4 \int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta$$

$$= \frac{M}{2\pi a^2}\cdot\pi a^4\cdot\frac{2}{3} + 2\cdot\frac{M}{2\pi a^2}\cdot a^4(-1)\left|\frac{\cos^3\theta}{3}\right|_0^{\pi/2}$$

$$= \frac{1}{3}Ma^2 + \frac{1}{3}Ma^2$$

$$I_{BD} = \frac{2}{3}Ma^2$$

$$\boxed{I_{BD} = \frac{2}{3}Ma^2} \longrightarrow \text{Result.}$$

(iii) <u>About Centroidal axis parallel to BD#</u>

The centroidal axis GC perpendicular to the axis of symmetry will be parallel to BD and at distance $OG = \frac{1}{2}a$. So by parallel axis theorem, we have

$$I_{BD} = I_{GC} + M \cdot OG^2$$

$$\tfrac{2}{3} Ma^2 = I_{GC} + M \cdot \frac{a^2}{4}$$

$$I_{GC} = \tfrac{2}{3} Ma^2 - \tfrac{1}{4} Ma^2$$

$$= Ma^2 \left(\tfrac{2}{3} - \tfrac{1}{4}\right)$$

$$\boxed{I_{GC} = \frac{5}{12} Ma^2} \longrightarrow \text{Result.}$$

(iv) About Tangent at the End of Symmetry Axis #

Tangent at the end A of symmetry axis is parallel to centroidal axis GC and at distance given by

$$GA = OA - OG$$

$$= a - \tfrac{1}{2}a = \tfrac{1}{2}a$$

By parallel axes theorem, we have

$$I_{tangent} = I_{GC} + M \cdot \left(\tfrac{1}{2}a\right)^2$$

$$= \frac{5}{12} Ma^2 + \frac{Ma^2}{4}$$

$$= Ma^2 \left(\frac{5}{12} + \frac{1}{4}\right)$$

$$= Ma^2 \left(\frac{5+3}{12}\right) = Ma^2 \cdot \frac{8}{12}$$

$$= Ma^2 \cdot \tfrac{2}{3}$$

$$\boxed{I_{tangent} = \tfrac{2}{3} Ma^2} \longrightarrow \text{Result.}$$

M.I of Uniform Solid Sphere

Question # Find the moment of inertia of a uniform Solid sphere about

(i) # a diameter (or axis of symmetry)

(ii) # a tangent at any point of sphere.

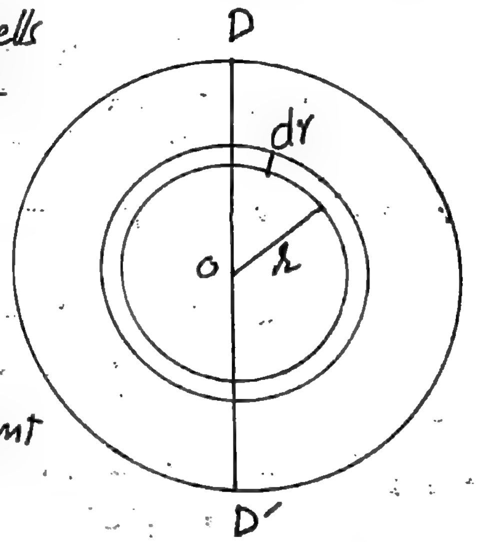
(* By M. Hussain Lecturer (Maths) Govt College Asghar Mall RWP)

Sol # **Method-I (Shell Method) #**

Consider a sphere of radius (a) & mass M. Let O be the centre of sphere and DD' diameter of the sphere.

Divide the sphere into Concentric spherical shells centred at the centre of the sphere.

Let r be the radius of one such shell with thickness dr.



Volume of the shell element

$= dV = \frac{4}{3}\pi r^3$

Area of the shell element $= 4\pi r^2$

Volume enclosed by the shell

$= 4\pi r^2 dr$

Mass of the shell $= dm = \rho(4\pi r^2 dr)$

where ρ is volume density of the sphere

Moment of inertia of shell about its diameter DD' (it will also be diameter of shell)

$dI_{DD'} = \frac{2}{3}(dm)\, r^2$ (by spherical shell Method)

$$dI_{DD'} = \rho \frac{2}{3}(4\pi r^2)\, dr \cdot r^2$$

$$= \rho \frac{8}{3} \pi r^4\, dr$$

Moment of inertia about diameter is

$$I_{DD'} = \rho \frac{8}{3} \pi \int_0^a r^4\, dr$$

$$= \rho \frac{8}{3} \pi \left[\frac{r^5}{5}\right]_0^a$$

$$= \rho \cdot \frac{8}{3} \pi \left[\frac{a^5}{5}\right]$$

Volume of sphere $= \frac{4}{3} \pi r^3$

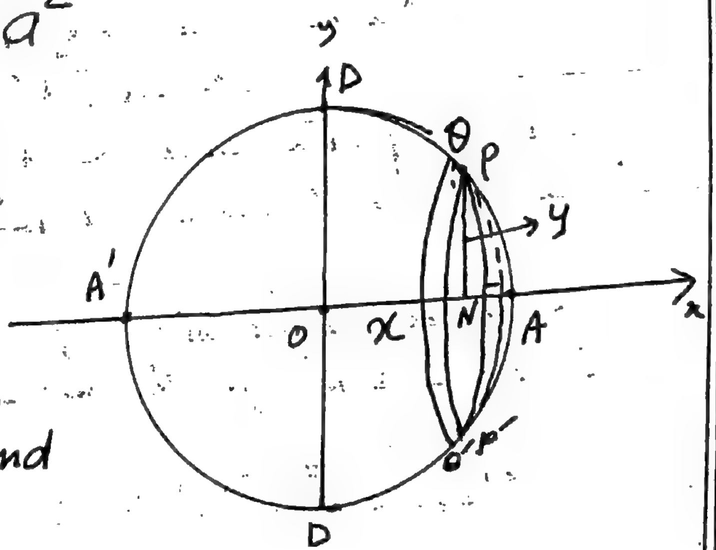
$$\rho = \frac{M}{\frac{4}{3}\pi a^3}$$

$$I_{DD'} = \frac{M}{\frac{4}{3}\pi a^3} \cdot \frac{8}{3} \pi \cdot \frac{a^5}{5}$$

$$I_{DD'} = \frac{2}{5} M a^2$$

Method - II #

Divide the sphere into circular slices (circular discs) all perpendicular to the diameter ox and centres at ox



Then $NP^2 = y^2 = a^2 - x^2$

Note equation of sphere is

$$x^2+y^2+z^2 = a^2$$

and circular disc is taken in xy-plane let dx be thickness of the disc. Then

Volum occupied by disc $= \pi y^2 dx$

Mass of the disc is

$$dm = \rho \pi y^2 dx$$

Moment of the disc about ox (centroidal axis perp to the plane of disc) is

$$dI_{ox} = \frac{1}{2}(dm)(\text{radius})^2$$

$$= \frac{1}{2}(\rho \pi y^2 dx)\cdot y^2$$

$$= \frac{1}{2}\rho \pi \, y^4 dx$$

$$= \frac{1}{2}\rho \pi (a^2-x^2)^2 dx$$

Hence M.I of the whole sphere about ox is

$$I_{ox} = \frac{1}{2}\rho \pi \int_{-a}^{a} (a^2-x^2)^2 dx$$

$$= \frac{1}{2}\rho \pi \cdot 2 \int_{0}^{a} (a^2-x^2)^2 dx$$

($\because$ integrand is even)

$$I_{ox} = \rho\pi \cdot \int_0^a (a^2 - x^2)^2 dx$$

let $x = a\sin\theta$
$dx = a\cos\theta \, d\theta$

When $x = 0$ $\quad \theta = 0$
$x = a$ $\quad \theta = \pi/2$

$$I_{ox} = \rho\pi \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^2 \cdot a\cos\theta \, d\theta$$

$$= \rho\pi a^5 \int_0^{\pi/2} \cos^5\theta \, d\theta$$

$$= \rho\pi a^5 \cdot \frac{4 \cdot 2}{5 \cdot 3}. \quad \text{By Wallis's formula}$$

$$\rho = \frac{M}{4/3 \, \pi a^3}$$

$$I_{ox} = \frac{3}{4}\frac{M}{\pi a^3} \cdot \pi \cdot a^5 \cdot \frac{4 \cdot 2}{5 \cdot 3}$$

$$I_{ox} = \frac{2}{5} M a^2$$

Method-III

$$I_{xx} = \rho \iiint (y^2 + z^2) \, dV$$

using spherical polar Co-ordinates

$x = r\sin\theta\cos\phi \qquad 0 \le \theta \le \pi$
$y = r\sin\theta\sin\phi \qquad 0 \le r \le a$
$z = r\cos\theta \qquad 0 \le \phi \le 2\pi$

$dV = r^2 \sin\theta \, dr\, d\theta\, d\phi$

$$I_{xx} = \rho \iiint (r^2 \sin^2\theta \cos^2\phi + r^2\cos^2\theta)\, dV$$

$$= \rho \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{0}^{a} (r^2 \sin^2\theta \cos^2\phi + r^2\cos^2\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \rho \int_0^a r^4\, dr \left[\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta \cos^2\phi\, d\theta\, d\phi + \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \cos^2\theta \sin\theta\, d\theta\, d\phi \right]$$

$$= \frac{\rho . a^5}{5}\left[4.2 \int_{\phi=0}^{\pi/2} \cos^2\phi\, d\phi . \int_0^{\pi/2} \sin^3\theta\, d\theta + |\phi|_0^{2\pi} (-1) \left| \frac{\cos^3\theta}{3} \right|_0^{\pi} \right]$$

$$= \frac{\rho a^5}{5}\left[8 . \frac{1}{2} . \frac{\pi}{2} . \frac{2}{3} + 2\pi(-1)\left[-1/3 - 1/3\right] \right]$$

$$= \frac{\rho a^5}{5}\left[\frac{4}{3}\pi + \frac{4\pi}{3} \right]$$

$$= \rho . \frac{8a^5\pi}{15}$$

$$= \frac{M}{\frac{4}{3}\pi a^3} . \frac{8a^5\pi}{15} = 2/5\, Ma^2$$

By Muhammad Hussain Lecturer (Maths) Govt College Asghar Mall RWP.

(ii) About Tangent #

Tanget to sphere will be parallel to diameter of the sphere.
Let LM be tangent of sphere and d, be diameter parallel to the (sphe) tangent.
Then by parallel axes Theorem, we have

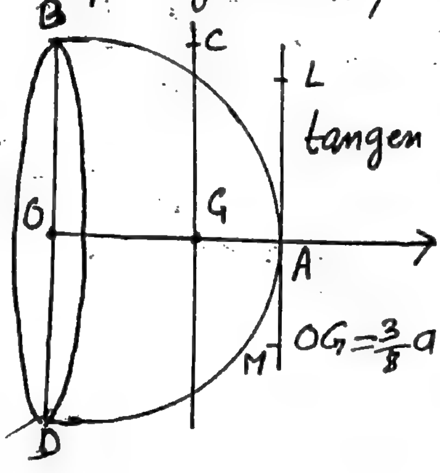
$$I_{LM} = I_d + Ma^2$$
$$= \frac{2}{5}Ma^2 + Ma^2$$
$$= \frac{7}{5}Ma^2$$

By M. Hussain Lecturer (Maths) Govt. College Asghar Mall.

M.I of Uniform Solid Hemi-sphere

Question# Find the moment of inertia of uniform hemi-sphere of Mass M, radius 'a' about
(I)# axis of symmetry
(ii)# a diameter of plane base
(iii)# an axis through its c-m and perpendicular to the axis of symmetry
(iv)# a tangent at the end of symmetry axis

Sol# Consider a uniform hemi-sphere of radius a, & mass M. Let G be its centre of mass and GC be centroidal axis perpendicular to the



symmetry axis. Let BD be the diameter of the plane base.

(i) **About Axis of Symmetry #**

Consider a uniform solid sphere whose centre is at O and whose radius is 'a'. Then

Mass of solid sphere $= 2$ (Mass of hemi-sphere)

$$= 2M$$

Moment of inertia of the sphere about OA is

$$\tfrac{2}{5}(2M)a^2$$

Now if we bisect the sphere by a plane through O perpendicular to OA, then contribution of each hemi-sphere is same towards the moment of inertia of the whole sphere.

Hence the moment of inertia of hemi-sphere is half of the m.I of whole sphere.

$$M.I \text{ of hemi-sphere about } OA = \frac{2}{5}Ma^2$$

(ii) Similarly moment of inertia of hemi-sphere about diameter BD is $\tfrac{2}{5}Ma^2$

(iii) **About Centroidal axis perp to OA #**

By parallel axes theorem, we have

$$I_{BD} = I_{GC} + M \cdot OG^2$$

$$\tfrac{2}{5}Ma^2 = I_{GC} + M\cdot\left(\tfrac{3}{8}a\right)^2$$

$$\tfrac{2}{5}Ma^2 = I_{GC} + \frac{9}{64}Ma^2$$

$$I_{GC} = \left(\frac{2}{5} - \frac{9}{64}\right) Ma^2$$

$$\boxed{I_{GC} = \left(\frac{83}{320}\right) Ma^2} \longrightarrow \text{Result.}$$

(iv) **About Tangent at End A #**

Tangent at the end A will be parallel to GC and at distance given by

$$GA = OA - OG$$
$$= a - \frac{3}{8}a$$
$$GA = \frac{5}{8}a$$

By parallel axes theorem.

$$I_{tangent} = I_{GC} + M\,GA^2$$
$$= \frac{83}{320} Ma^2 + M \cdot \frac{25}{64} a^2$$
$$= Ma^2\left(\frac{83}{320} + \frac{25}{64}\right)$$
$$= Ma^2\left(\frac{83 + 125}{320}\right)$$
$$= Ma^2\left(\frac{\overset{26}{\cancel{208}}}{\underset{40}{\cancel{320}}}\right) = \frac{13}{20} Ma^2$$

$$\boxed{I_{tangent} = \frac{13}{20} Ma^2} \longrightarrow \text{Result.}$$

By Muhammad Hussain Lecturer (Maths) Govt. College Asghar Mall Rawalpindi.

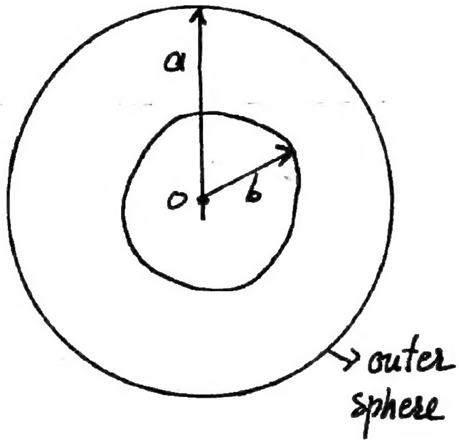
Question # Show that the moment of inertia of a hollow sphere, whose external and internal radii are 'a' and 'b' about a diameter axis are given by

$$\frac{2M}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}$$

OR

Find the moment of inertia of a spherical shell bounded by two concentric spheres of radii a & b respectively, about diameter.

Sol # The hollow sphere of external and internal radii 'a' & 'b' will be region bounded by two concentric solid spheres of radii 'a' & 'b'.



Mass of the sphere of radius 'a' is

$$M_a = \frac{4}{3}\pi a^3 \rho$$

Mass of the sphere of radius 'b' is

$$M_b = \frac{4}{3}\pi b^3 \rho$$

Let M be the mass of hollow sphere. Then

$$M = M_a - M_b = \frac{4}{3}\pi \rho (a^3 - b^3) \rightarrow (1)$$

M.I of sphere of radius 'a' about diameter is

$$I_a = \frac{1}{5} M_a a^2 = \frac{1}{5} \cdot \left(\frac{4}{3}\pi a^3 \rho\right) a^2$$

$$= \frac{4}{15} \pi a^5 \rho$$

Similarly $I_b = \frac{4}{15} \pi b^5 \rho$

Let I be the moment of inertia of hollow sphere.

$$I = I_a - I_b$$

$$= \frac{4}{15}\pi\rho\,(a^5 - b^5)$$

But $\rho = \dfrac{3M}{4\pi(a^3 - b^3)}$ by ①

$$\Rightarrow \quad I = \frac{1}{5}\,\frac{M(a^5 - b^5)}{a^3 - b^3}$$

which is the required result.